On some generalization of the notion of power index: ordinality and criticality in voting games

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Outline

Basic-Basic on coalitional games

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Given a game, a probabilistic value may be computed to convert information about the worth that coalitions can achieve into a personal attribution (of payoff) to each of the players:

$$\pi_i^p(v) = \sum_{S \in 2^{N \setminus \{i\}}} p^i(S)\big(\, v(S \cup \{i\}) - v(S)\big)$$

for each $i \in N$, and where $p = (p^i : 2^{N \setminus \{i\}} \to \mathbb{R}^+)_{i \in N}$, is a collection of non negative real functions fulfilling the condition $\sum_{S \in 2^{N \setminus \{i\}}} p^i(S) = 1$.

Semivalues

A particular interesting case are semivalues (see Dubey et al. 1981, Carreras and Freixas 1999; 2000):

$$\pi_i^{\mathbf{p}}(v) = \sum_{S \subset N : i \notin S} p_s \big(\ v(S \cup \{i\}) - v(S) \big)$$

for each $i \in N$, where p_s represents the probability that a coalition $S \in 2^N$ (of cardinality s) with $i \notin S$ forms. So coalitions of the same size have the same probability to form!

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(of course $\sum_{s=0}^{n-1} \binom{n-1}{s} p_s = 1$).

Shapley and Banzhaf (regular) semivalues

- The Shapley value (Shapley 1953) is a regular semivalue $\pi^{\mathbf{p}^{\phi}}(v)$ with

$$\mathbf{p}_s^{\phi} = \frac{1}{n\binom{n-1}{s}} = \frac{s!(n-s-1)!}{n!}$$

for each $s = 0, 1, \ldots, n-1$ (i.e., the cardinality is selected with the same probability).

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- The *Banzhaf value* (Banzhaf III 1964), which is defined as the regular semivalue $\pi^{\mathbf{p}^{\beta}}(v)$ with

$$\mathbf{p}_s^{\beta} = \frac{1}{2^{n-1}}$$

for each $s = 0, 1, \ldots, n-1$, (i.e., each coalition has an equal probability to be chosen)

many applications...

- Cost allocation problems (Littlechild and Thompson 1977, Young 1994, Fragnelli et al. 1999)

- Social interaction (Myerson 1977, Gómez et al. 2003)

- Water related issues (Loehman and Whinston 1976, Dinar et al. 1986)

- Epidemiology and risk analysis (Cox 1985, Land and Gefeller 1997)

- Computational biology (Moretti et al. 2007, Kaufman 2004)

- Reliability theory (Ramamurthy 1990), theory of belief functions (Smets 1990), etc... (see also the survey by Moretti and Patrone (2008)).

Classical assumptions revised

- In cooperative game theory, classical measures of agents' power, like the Shapley index (Shapley and Shubik (1954)) or the Banzhaf index (Banzhaf (1964)), are computed on the characteristic function of a game

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- In practical situations, however, the information concerning both the power and the effective cooperation possibilities of coalitions may concern hardly quantifiable factors like bargaining abilities, moral and ethical codes and other "psychological" attributes (Maschler (1963))

- In addition to what it can gain by itself, a coalition may obtain some more "power" by threatening not to cooperate with other players and causing them losses

Social ranking problem

A company with three employees $1, 2$ and 3 working in the same department. According to the opinion of the manager of the company, the job performance of the different teams $S \subseteq N = \{1, 2, 3\}$ is as follows:
$\{1, 2, 3\} \succ \{3\} \succ \{2\} \succ \{1, 3\} \succ \{2, 3\} \succ \{1\} \succ \{1, 2\} \succ \emptyset$.

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Can we state more precisely the reasons driving us to this conclusion? and who between 1 and 2 is more "productive"?

First attempt: π^p-invariant total preorders

Given a total preorder $\succcurlyeq$ on 2^N, we denote by $V(\succcurlyeq)$ the class of coalitional games that numerically represent $\succcurlyeq$ (for each $S, V \in 2^N$, $S \succcurlyeq V \Leftrightarrow u(S) \geq u(V)$ for each $u \in V(\succcurlyeq)$).

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DEF. Let $\pi^{\mathbf{p}}$ be a regular semivalue. A total prorder $\succcurlyeq$ on 2^N is *π^p-invariant* iff there exists $R \in \mathcal{T}^N$ such that for each numerical representation $v \in V(\succcurlyeq)$ we have that

$$iRj \Leftrightarrow \pi_i^p(v) \geq \pi_j^p(v)$$

for all $i, j \in N$. ∎

Example: a Shapley-invariant total preorder

Let $N = \{1, 2, 3\}$ and let $\succcurlyeq$ be a total preorder on N such that
$\{1, 2, 3\} \succ \{3\} \succ \{2\} \succ \{1, 3\} \succ \{2, 3\} \succ \{1\} \succ \{1, 2\} \succ \emptyset$.
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For every $v \in V(\succcurlyeq)$, the difference in terms of Shapley value $\phi(v)$ is

$$\phi_2(v) - \phi_1(v) = \frac{1}{2}\big(v(2) - v(1)\big) + \frac{1}{2}\big(v(2,3) - v(1,3)\big) > 0$$

On the other hand

$$\phi_3(v) - \phi_2(v) = \frac{1}{2}\big(v(3) - v(2)\big) + \frac{1}{2}\big(v(1,3) - v(1,2)\big) > 0.$$

... $\pi^{\mathbf{p}}$-invariant for other regular semivalues

Note that $\succcurlyeq^a$ is $\pi^{\mathbf{p}}$-invariant for every regular semivalue such that $p_0 \geq p_2$:

$$\pi_2^{\mathbf{p}}(v) - \pi_1^{\mathbf{p}}(v) = (p_0 + p_1)\big(v(2) - v(1)\big) + (p_1 + p_2)\big(v(2,3) - v(1,3)\big) > 0$$

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for every $v \in V(\succcurlyeq^a)$.

Total preorder $\pi^{\mathbf{p}}$-invariant for no regular semivalues

It is quite possible that for a given preorder there is no $\pi^{\mathbf{p}}$-invariant semivalue associated to it. It is enough, for instance, to consider the case $N = \{1, 2, 3\}$ and the following total preorder:

$$N \succ \{1, 2\} \succ \{2, 3\} \succ \{1\} \succ \{1, 3\} \succ \{2\} \succ \{3\} \succ \emptyset.$$

Then it is easy to see that 1 and 2 cannot be ordered since, fixed a semivalue $\mathbf{p}$ the quantity

$$\pi_2^{\mathbf{p}}(v) - \pi_1^{\mathbf{p}}(v) = (p_0 + p_1)(v(\{1\}) - v(\{2\})) + (p_1 + p_2)(v(\{1, 3\}) - v(\{2, 3\})$$

can be made both positive and negative by suitable choices of v.

Formal problem

$N = \{1, \ldots, n\}$ and 2^N is the powerset of N. $\mathcal{T}^N$ and $\mathcal{T}^{2^N}$ are the set of all total preorders on N and on 2^N, respectively.

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A total preorder $\succcurlyeq \in \mathcal{T}^{2^N}$ is said a *power relation*, that is, for each $S, T \in 2^N$, $S \succcurlyeq T$ stands for 'S is considered at least as strong as T according to the power relation $\succcurlyeq$'.

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We call a map $\rho : \mathcal{T}^{2^N} \longrightarrow \mathcal{T}^N$, assigning to each power relation in $\mathcal{T}^{2^N}$ a total preorder on N, a *social ranking solution* or, simply, a *social ranking*.

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Precisely, for each $i, j \in N$, $i\rho(\succcurlyeq)j$ stands for 'i is considered at least as influential as j according to the social ranking $\rho(\succcurlyeq)$'.

Characterizing total p-aligned total pre-orders

Power of threatening: if player i has more possibilities than j to form coalitions at least as powerful as S, for every possible coalition $S \in 2^N$, we say that i dominates j

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Axiom [DOM]

- We say that a social ranking satisfies DOM iff

$$i \text{ dominates } j \Rightarrow i \rho(\succcurlyeq) j$$

Example

Consider again the coalitional power relation

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	$\{1,2,3\}$	$\{3\}$	$\{2\}$	$\{1,3\}$	$\{2,3\}$	$\{1\}$	$\{1,2\}$	$\emptyset$
player 1	1	1	1	2	2	3	4	4
player 2	1	1	2	2	3	3	4	4
player 3	1	2	2	3	4	4	4	4

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player 3	1	2	2	3	4	4	4	4

Note that 3 (strictly) dominates both 1 and 2 and 2 (strictly) dominates 1. So if ρ satisfies the DOM axiom we have $3\rho(\succcurlyeq)2\rho(\succcurlyeq)$ (strictly)

Connections with the Banzhaf power index

Theorem

Let $\succcurlyeq \in \mathcal{P}^{2^N}$ *and For each* $i, j \in N$

$$i \text{ dominates } j \Leftrightarrow [\beta_i(v) \geq \beta_j(v) \text{ for every } v \in V(\succcurlyeq)].$$

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A social ranking that satisfies the DOM property is such that if i has more power than j according the Banzhaf power index of every game representing $\succcurlyeq$, then i is ranked stronger than j.

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Similar results for other semivalues (using alternative versions of dominance, weighted according to the probability of coalition formation... see Lucchetti, Moretti and Patrone (2015) and Moretti (2015)).

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Note that both 1 and 2 dominate 3, whereas neither 1 dominates 2 nor 2 dominates 1.

Axiom [Responsiveness, RESP] A total preorder $\succcurlyeq$ on 2^N satisfies the *responsiveness* property iff for all $A \in 2^N \setminus \{N, \emptyset\}$, for all $x \in A$ and for all $y \in N \setminus A$ the following conditions holds

$$A \succcurlyeq (A \setminus \{x\}) \cup \{y\} \Leftrightarrow \{x\} \succcurlyeq \{y\}$$

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This axiom was introduced by Roth (1985) studying colleges' preferences for the "college admission problem" (see also Gale and Shapley (1962)).

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- Bossert (1995) used the same property for ranking sets of alternatives with a fixed cardinality and to characterize the class of *rank-ordered lexicographic* extensions.

Equivalent formulation of Responsiveness

Let $\succcurlyeq$ be a binary relation on N. A binary relation $\succcurlyeq$ on 2^N satisfies the *responsiveness* property on 2^N (and with respect to $\succcurlyeq$) iff for all $i, j \in N$ and all $S \in 2^N$, $S \cap \{i, j\} = \emptyset$ we have that

$$S \cup \{i\} \succcurlyeq S \cup \{j\} \Leftrightarrow \{i\} \succcurlyeq \{j\}.$$

RESP-Dominance

If coalition $S \cup \{i\}$ is stronger than coalition $S \cup \{i\}$ for each S not containing neither i nor j, then individual i should be ranked higher than a individual j.

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i r-dominates j in $\succcurlyeq$ if $S \cup \{i\} \succcurlyeq S \cup \{j\}$ for each $S \in 2^{N \setminus \{i,j\}}$ (we also say that *i strictly r-dominates j* in $\succcurlyeq$ if i r-dominates j and in addition there exists $S \in 2^{N \setminus \{i,j\}}$ such that $\neg(S \cup \{j\} \succcurlyeq S \cup \{i\})$).

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Axiom [R-DOM] A social ranking $\rho : \mathcal{T}^{2^N} \longrightarrow \mathcal{T}^N$ satisfies the R-DOM property on $\mathcal{T}^{2^N} \subseteq \mathcal{T}^{2^N}$ iff for all $\succcurlyeq \in \mathcal{C}^{2^N}$ and $i, j \in N$,
if i r-dominates j in $\succcurlyeq$ then $i \rho(\succcurlyeq) j$
[and $\neg(j \rho(\succcurlyeq) i)$ if i strictly r-dominates j in $\succcurlyeq$].

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1 vs. 2
$1 \sim 2$
$13 \succ 23$
$14 \sim 24$
$134 \sim 234$

By the R-DOM property, we should have $1\rho(\succcurlyeq)2$ and $\neg(2\rho(\succcurlyeq)1)$.

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Axiom[ANON] A social ranking $\rho : \mathcal{T}^{2^N} \longrightarrow \mathcal{T}^N$ satisfies the *anonymity* (ANON) property on $\mathcal{T}^{2^N} \subseteq \mathcal{T}^{2^N}$ iff

$$i \rho(\succcurlyeq) j \Leftrightarrow \pi(i) \rho(\succcurlyeq) \pi(j)$$

for all $i, j \in N$, $\pi \in \Pi$ and $\succcurlyeq \in \mathcal{C}^{2^N}$ such that for each $S \in 2^{N \setminus \{i,j\}}$

$$S \cup \{i\} \succcurlyeq S \cup \{j\} \Leftrightarrow \pi(S) \cup \{\pi(i)\} \succcurlyeq \pi(S) \cup \{\pi(j)\}.$$

Π is the set of all bijections $\pi : N \to N$. With a slightly abuse of notations, we also denote by $\pi(S)$ the image under π of a coalition S, i.e. $\pi(S) = \{\pi(i) : i \in S\}$.

Example ANON

2 vs. 3
$2 \sim 3$
$12 \prec 13$
$24 \succ 34$
$124 \sim 134$

Take $\pi(1) = 4, \pi(2) = 3, \pi(3) = 2$ and $\pi(4) = 1$. Then by ANON, we should have $2\rho(\succcurlyeq)3 \Leftrightarrow 3\rho(\succcurlyeq)2$.

Incompatibility ($|N| > 3$)

Prop. There is no social ranking rule $\rho : \mathcal{T}^{2^N} \longrightarrow \mathcal{T}^N$ which satisfies R-DOM and ANON on $\mathcal{T}^{2^N}$ ($|N| > 3$).

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Prop. There is no social ranking rule $\rho : \mathcal{T}^{2^N} \longrightarrow \mathcal{T}^N$ which satisfies R-DOM and ANON on $\mathcal{T}^{2^N}$ ($|N| > 3$).

Consider a power relation $\succcurlyeq \in \mathcal{T}^{2^N}$ with $N = \{1, 2, 3, 4\}$ and such that

1 vs. 2	2 vs. 3	1 vs. 3
$1 \sim 2$	$2 \sim 3$	$1 \sim 3$
$13 \succ 23$	$12 \prec 13$	$12 \prec 23$
$14 \sim 24$	$24 \succ 34$	$14 \succ 34$
$134 \sim 234$	$124 \sim 134$	$124 \sim 234$

ANON (twice) implies that $2\rho(\succcurlyeq)3$, $3\rho(\succcurlyeq)2$ and $1\rho(\succcurlyeq)3$, $3\rho(\succcurlyeq)1$. By the R-DOM property, we should have $1\rho(\succcurlyeq)2$, and $\neg(2\rho(\succcurlyeq)1)$, which yields a contradiction with the transitivity of $\rho(\succcurlyeq)$ (see Moretti and Ozturk (2016)).

Some further notations

Suppose we have a ranking $\succcurlyeq \in \mathcal{T}^{2^N}$ of the form

$$S_1 \succcurlyeq S_2 \succcurlyeq S_3 \succcurlyeq \cdots \succcurlyeq S_{2^n}.$$

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Unless the ranking is a complete order, some indifferences are present. Thus we write

$$\Sigma_1 \succ \Sigma_2 \succ \Sigma_3 \succ \cdots \succ \Sigma_l$$

to denote the same ranking, but among the equivalence classes with respect to $\sim$.

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For example, such equivalence classes could represent the levels of scientific productivity reached by different groups of researchers.

Some notations

For any element $x \in N$, denote by x_k the number of sets containing x in Σ_k, that is

$$x_k = |\{S \in \Sigma_k : x \in S\}|$$

for $k = 1, \ldots, l$. Let $\theta_{\succeq}(x)$ be the l-dimensional vector $\theta_{\succeq}(x) = (x_1, \ldots, x_l)$ associated to $\succeq$.

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Now consider the lexicographic order among vectors:

$$\mathbf{x} \geq_L \mathbf{y} \quad \text{if either } \mathbf{x} = \mathbf{y} \quad \text{or} \quad \exists j : x_i = y_i, i = 1, \ldots, j-1 \wedge x_j > y_j.$$

Excellence ranking function

Definition (Bernardi, Luchetti, Moretti (2017))

The *excellence ranking function* is the function $\rho : \mathcal{T}^{2^N} \longrightarrow \mathcal{T}^N$ defined in the following way on $\succcurlyeq$:

$$x \rho(\succcurlyeq) y \qquad \text{if} \qquad \theta_{\succeq}(x) \;\; \geq_L \;\; \theta_{\succeq}(y).$$

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when $\succeq$ is a complete order, then the $(2^n - 1)$-dimensional vector $\theta_{\succeq}(i)$ is boolean, i.e. made by only zeros and ones.

Example

Consider the coalitional power relation

$$\{1,2,3\} \succ \{2\} \succ \{1,3\} \succ \{1,2\} \succ \{3\} \succ \{1\} \succ \emptyset \succ \{2,3\}$$

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Σ_k	$\{1,2,3\}$	$\{2\}$	$\{1,3\}$	$\{1,2\}$	$\{3\}$	$\{1\}$	$\emptyset$	$\{2,3\}$
$\theta_{\succeq}(1)$	1	0	1	1	0	1	0	0
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So the excellence ranking gives $2\rho(\succcurlyeq)1\rho(\succcurlyeq)3$.

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So the excellence ranking gives $1\rho(\succcurlyeq)2\rho(\succcurlyeq)3$.

Orders of criticality in voting games

We consider a Parliament that produced a majority coalition. If the majority corresponds to a minimal winning coalition all the parties result critical, i.e. each of them is able to destroy the majority when leaving,

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If the majority is formed by a quasi-minimal winning coalition including five parties, only one of which is critical, each non-critical party may reclaim some power from the unique critical party for its role in keeping other parties non-critical.

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We can say that the critical parties have a first order of criticality, while the non-critical ones have a higher order of criticality.

Second orders of criticality

Definition

Let $M \subseteq N$, with $|M| \geq 3$, be a winning coalition; let $i \in M$ be a player s.t. $v(M \setminus \{i\}) = 1$. We say that player i is *second order critical (soc) for coalition* M*, via player* $j \in M \setminus \{i\}$ iff $v(M \setminus \{i, j\}) = 0$ with $v(M \setminus \{j\}) = 1$.

Example

Consider the weighted majority situation [51; 40, 8, 5, 5, 5]; the first party is the unique critical one, while the other four parties are critical of the second order, even if the last three parties are critical only via the second party.

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Of course you can generalize this definition to further orders of criticality... (Dall'Aglio, Fragnelli, Moretti (2016))

An example from the Italian Senate

During the eighties, the Italian governments included five parties: namely Christian Democracy (Democrazia Cristiana), Italian Socialist Party (Partito Socialista Italiano), Italian Social-Democratic Party (Partito Socialista Democratico Italiano), Italian Republican Party (Partito Repubblicano Italiano), Italian Liberal Party (Partito Liberale Italiano).

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Party	seats
Democrazia Cristiana (DC)	145
Partito Socialista Italiano (PSI)	32
Partito Socialdemocratico Italiano (PSDI)	9
Partito Repubblicano Italiano (PRI)	6
Partito Liberale Italiano (PLI)	2

The quota to have the majority was 162.

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In 1983 the PSI threatened to leave the five-party alliance unless Bettino Craxi, the PSI party's leader, was made Prime Minister. The DC party accepted this compromise.

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In 1983 the PSI threatened to leave the five-party alliance unless Bettino Craxi, the PSI party's leader, was made Prime Minister. The DC party accepted this compromise.

Maybe, the DC party had evaluated the threat of the PSI as credible in view of the fact the the PSI was SOC for the coalition DC, PSI, PSDI, PRI, PLI, via all the other non-critical parties (specifically, PSI vs. PSDI, PSI vs. PRI and PSI vs. PLI), whereas all the other pairs of non-critical parties are not in a SOC relation.

SOC power index for monotonic games

We want to compute the probability a player is SOC for some coalitions in (N, v) (a monotonic simple game: $v(S) \in \{0, 1\}$ for each $S \subseteq N$, $v(N) = 1$, and such that if $v(S) = 1$ then $v(T) = 1$ for all $T \supseteq S$).

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Take $S \in 2^{N \setminus \{i,j\}}$ with $v(S \cup \{i, j\}) = 1$ and define $C_{ij}(S)$ as

$$C_{ij}(S) = \min\{v(S \cup \{i\}), v(S \cup \{j\})\} - v(S).$$

By monotonicity of v we have four possible cases:

	$v(S \cup \{i\})$	$v(S \cup \{j\})$	$v(S)$	$C_{ij}(S)$
1)	0	1	0	0
2)	1	0	0	0
3)	1	1	1	0
4)	1	1	0	1

The only case in which i is SOC for $S \cup \{i, j\}$ via j is the last one and $C_{ij}(S) = 1$. Note also that, in general, $C_{ij}(S) = C_{ji}(S)$.

SOC power index for monotonic games

Let $\mathbf{p} = (p_0, \ldots, p_{n-1})$ be a probability vector.

$$\Gamma_{ij}^{\mathbf{p}}(v) = \sum_{S \in 2^{N \setminus \{i,j\}}} p_{s+1} C_{ij}(S).$$

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Example

Consider again the the Italian Seante during the Eighties. This leads to the weighted majority situation $[162; 145, 32, 9, 6, 2]$ on the player set $\{DC, PSI, PSDI, PRI, PLI\}$. There are two minimal winning coalitions: $\{\{DC, PSI\}, \{DC, PSDI, PRI, PLI\}\}$.

	DC	PSI	PSDI	PRI	PLI
$\pi^{\mathbf{p}^\beta}$	$\frac{9}{16}$	$\frac{7}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$
$C^{\mathbf{p}^\beta}$	0	$\frac{3}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$

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