Basic (mathematical) knowledge on the structure of voting systems with several levels of approval

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Outline

Tripartitions:

$$(S_1, S_2, S_3) \in 3^N$$

- S_1 set of "yes" voters
- S_2 set of "abstainers"
- S_3 set of "no" voters

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Monotonicity:

$$(S_1, S_2, S_3) \sqsubseteq (T_1, T_2, T_3)$$

if

$$S_1 \subseteq T_1 \quad \text{and} \quad S_2 \subseteq T_1 \cup T_2$$

A $(3,2)$ simple game (N,V) consists of a finite set N of voters together with a value function $V : 3^N \longrightarrow \{win, \ lose\}$ such that:

- $(N,\emptyset,\emptyset)$ wins,
- $(\emptyset,\emptyset,N)$ loses,
- $S \sqsubseteq T$ and $V(S) =$ wins implies $V(T) =$ wins.

Example

Consider the (3, 2) game defined on $N = \{1, 2, 3\}$ by its set of minimal winning tripartitions
$W^m = \{(12, \emptyset, 3), (1, 23, \emptyset), (23, 1, \emptyset)\}$ (keys and commas omitted for tripartitions).

$(123, \emptyset, \emptyset)$	winning but non-minimal
$(12, 3, \emptyset)$	winning but non-minimal
$(13, 2, \emptyset)$	winning but non-minimal
$(1, 23, \emptyset)$	minimal winning
$(12, \emptyset, 3)$	minimal winning
$(23, 1, \emptyset)$	minimal winning
others	losing

A $(3,2)$ simple game (N, W) is a weighted $(3,2)$ game if there exists a weight function $w : N \to \mathbb{R}^3$ and a quota q such that

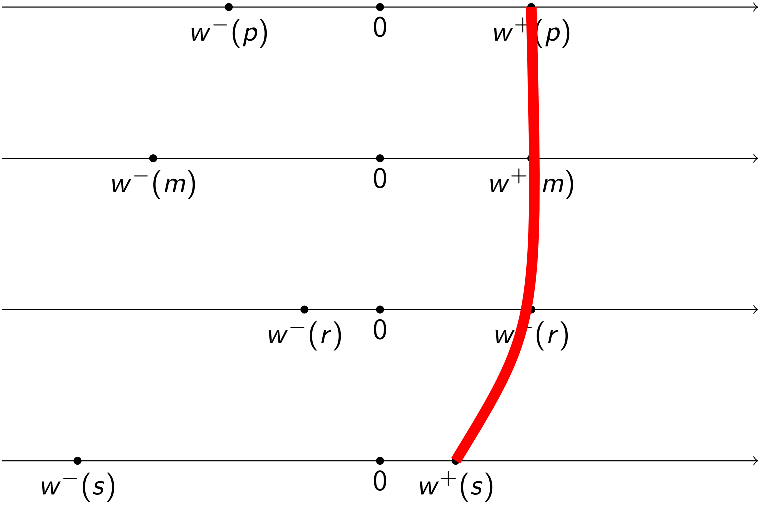
$$S \in W \iff w(S) \geq q$$

where $w(S) = \sum_{p \in S_1} w^+(p) + \sum_{p \in S_3} w^-(p)$ for all $S = (S_1, S_2, S_3) \in 3^N$.

The only requirement for weights is:

$$w^+(p) \geq 0 \geq w^-(p)$$

Some weights in a weighted game



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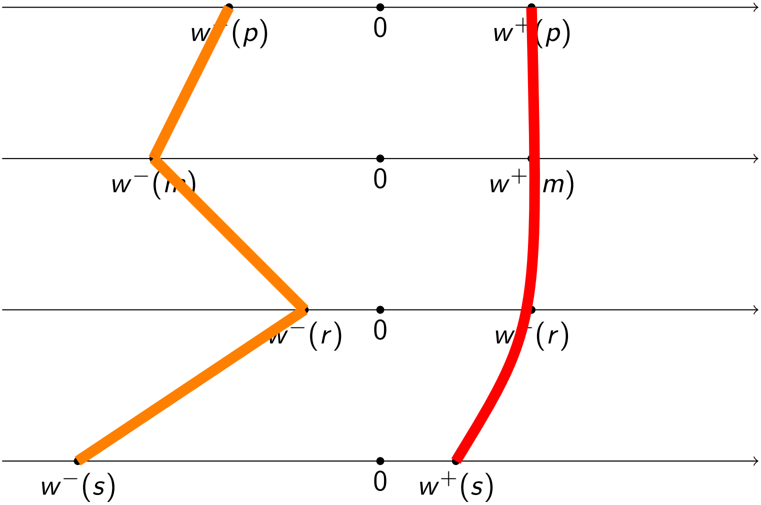

Example

A resolution is carried in the Security Council if at least nine members support it and no permanent member is explicitly opposed. Let $P = \{1, 2, 3, 4, 5\}$ and $R = \{6, 7, ..., 15\}$ be resp. the set of permanent and nonpermanent members, and

$$V(S) = V(S_1, S_2, S_3) = \begin{cases} \text{win} & \text{if } |S_1| \geq 9 \text{ and } S_3 \cap P = \emptyset \\ \text{lose} & \text{otherwise} \end{cases}$$

This voting system with abstention can be represented as

$$[\ 9;\ \underbrace{(1, 0, -6), ..., (1, 0, -6)}_{5}, \underbrace{(1, 0, 0), ..., (1, 0, 0)}_{10}\ ]$$

Example

A juror has three choices–convict, acquit or be neutral–but the outcome of the jury as a whole has only two choices–convict or acquit. Assume there are 4 jurors, A, B, C, and D.
The collective decision is "convict" *if and only if*:

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Version 1 more jurors vote for "convicting" than for "acquittal".

Version 2 at least one juror votes for "convicting" and at most one juror votes for "acquittal".

Version 3 more jurors vote for "convicting" than for "acquittal" and juror A votes for "convicting".

Example

Versions 1 and 2 correspond to anonymous $(3,2)$ games. In version 3, juror A plays a distinguished role.

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Voting rule in **version 1** is weighted, with $w(p) = (1, 0, -1)$ for each $p \in N$ and $q = 1$.

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Voting rule in **version 3** is weighted. Two representations:

$$[\ 4;\ (4, 0, 0), (1, 0, -1), (1, 0, -1), (1, 0, -1)\ ]$$

$$[\ 4;\ (4, 0, -1), (1, 0, -1), (1, 0, -1), (1, 0, -1)\ ]$$

Two consecutive stronger conditions of a weighted $(3,2)$:

A strongly weighted $(3,2)$ game is a weighted $(3,2)$ game that admits a representation such that for every pair of voters p and r, either

$$w^{+}(p) \geq w^{+}(r) \quad \text{and} \quad -w^{-}(p) \geq -w^{-}(r)$$

or

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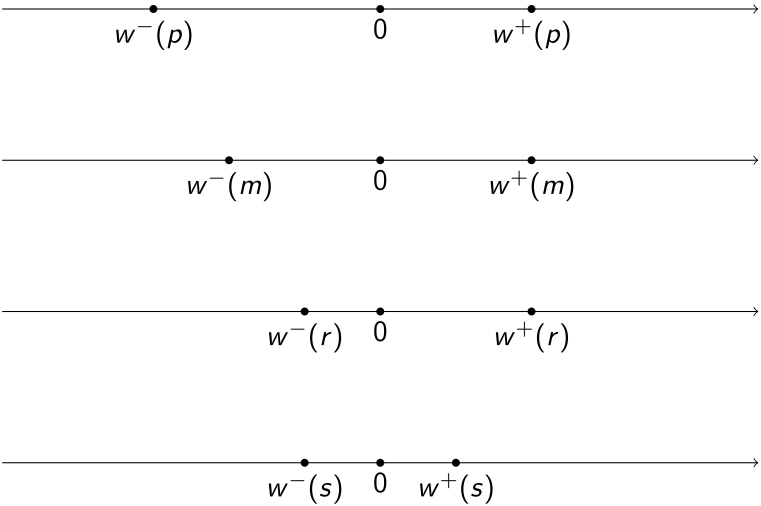
A zero-centered strongly weighted $(3,2)$ game is a strongly weighted $(3,2)$ game that admits a representation with weights $w^{+}(p) = -w^{-}(p)$ for each voter $p \in N$.

Weights in a strongly weighted game



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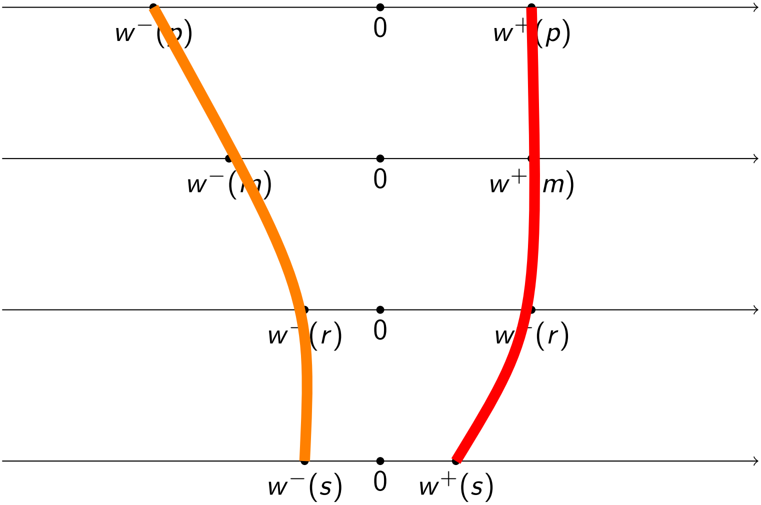
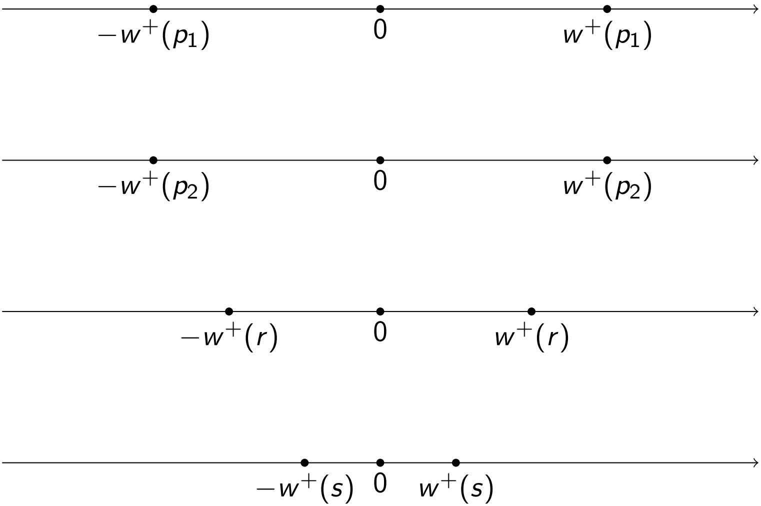
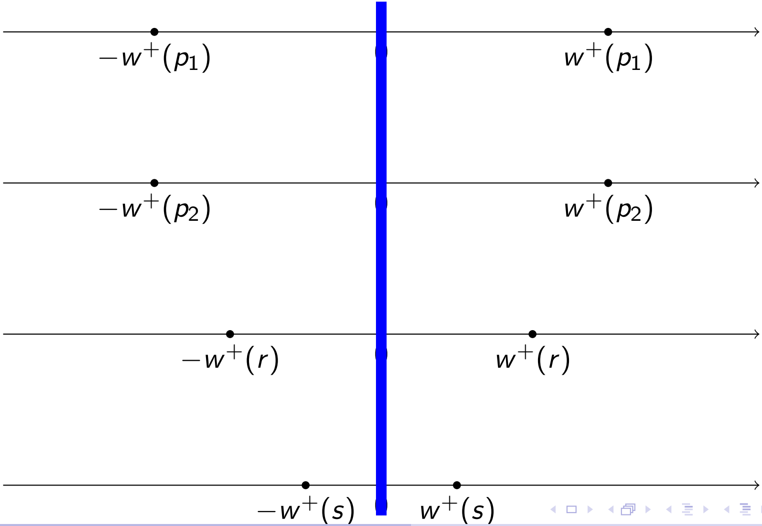


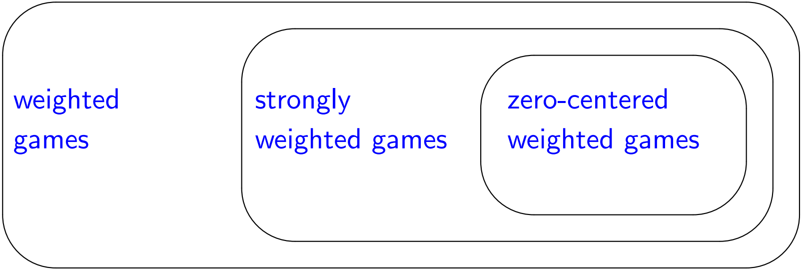
Weights in a zero-centered strongly weighted game



Weights in a zero-centered strongly weighted game



Inclusion of weighted VRA



Examples revisited

- The 4 jurors example in version 1 is a zero-centered strongly weighted (3,2) game. $[\ 1;\ \underbrace{(1,0,-1),\ldots,(1,0,-1)}_{4}\ ]$

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- The 4 jurors example in version 3 is a strongly weighted (3,2) game. $[\ 4;\ (4,0,-1),(1,0,-1),(1,0,-1),(1,0,-1)\ ]$

Examples revisited

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- The UNSC voting system with abstention is a strongly weighted (3,2) game.
$[\ 9;\ \underbrace{(1,0,-6),\ldots,(1,0,-6)}_{5},\underbrace{(1,0,0),\ldots,(1,0,0)}_{10}\ ]$

Tchantcho et. al. (2008). The I-influence relation.

Let (N, V) be a $(3,2)$ game, $p, r \in N$. Voter p is said to be at least as *influential* as r, $p \succsim_I r$, if for all $(S_1, S_2, S_3) \in 3^N$ it yields:

- $V(S_1 \cup p, S_2 \setminus p, S_3) \geq V(S_1 \cup r, S_2 \setminus r, S_3)$ if $p, r \in S_2$,
- $V(S_1, S_2 \cup p, S_3 \setminus p) \geq V(S_1, S_2 \cup r, S_3 \setminus r)$ if $p, r \in S_3$, and
- $V(S_1 \cup p, S_2, S_3 \setminus p) \geq V(S_1 \cup r, S_2, S_3 \setminus r)$ if $p, r \in S_3$.

$$(S_1 \cup p, S_2 \setminus p, S_3)$$

$$(S_1, S_2 \cup p, S_3 \setminus p)$$

$$(S_1 \cup p, S_2, S_3 \setminus p)$$

A weighted game non being I-complete

Example

Consider the $(3,2)$ simple game with set of voters $N = \{a, b, c\}$:

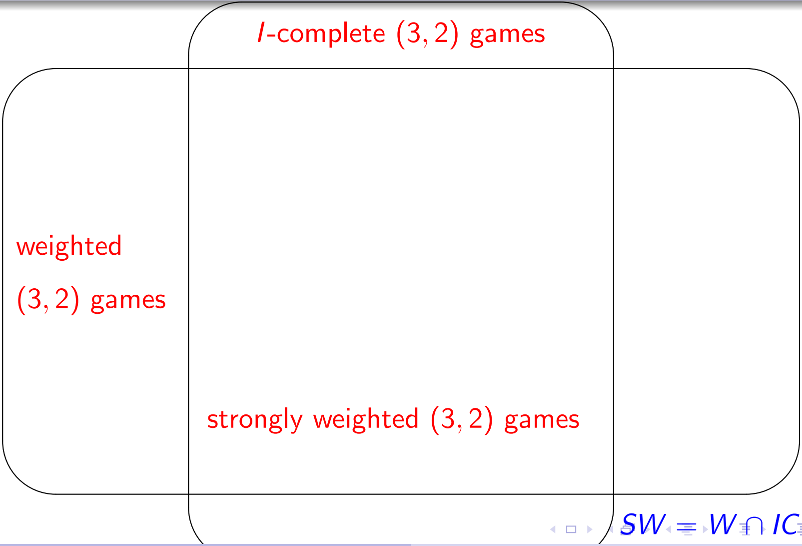
$$W^m = \{(\emptyset, bc, a), (a, c, b)\} \quad \mapsto \quad L^M = \{(a, b, c), (c, a, b)\}$$

It is weighted

$$q = 0; \quad w(a) = (1, 0, 0), \quad w(b) = (0, 0, -1), \quad w(c) = (0, 0, -2).$$

but not I-complete: $a \not\succsim_I b$ and $b \not\succsim_I a$, thus the game is not I-complete.

Weighted and I-complete (3,2) games



$$SW = W \cap IC$$

1. Voting rules with abstention or $(3,2)$ simple games
2. Weighted $(3,2)$ games
3. Complete $(3,2)$ games
4. Linking weightedness and completeness
5. A little bit on (Banzhaf) power: two versions

(i) D^+-desirability.

$$p \succsim_{D^+} r \iff V(S_1 \cup p, S_2 \setminus p, S_3) \geq V(S_1 \cup r, S_2 \setminus r, S_3)$$

$$(S_1 \cup p, S_2 \setminus p, S_3)$$

(ii) D^--desirability.

$$p \succsim_{D^-} r \iff V(S_1, S_2 \cup p, S_3 \setminus p) \geq V(S_1, S_2 \cup r, S_3 \setminus r)$$

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(iii) $D^{\pm}$-desirability.

$$p \succsim_{D^{\pm}} r \iff V(S_1 \cup p, S_2, S_3 \setminus p) \geq V(S_1 \cup r, S_2, S_3 \setminus r)$$

$$(S_1 \cup p, S_2, S_3 \setminus p)$$

(*i*) A $(3,2)$ game is D^+-complete *if* either $p \succsim_{D^+} r$ or $r \succsim_{D^+} p$ for all pair p, r of voters.

(*ii*) A $(3,2)$ game is D^--complete *if* either $p \succsim_{D^-} r$ or $r \succsim_{D^-} p$ for all pair p, r of voters.

(*iii*) A $(3,2)$ game is $D^\pm$-complete *if* either $p \succsim_{D^\pm} r$ or $r \succsim_{D^\pm} p$ for all pair p, r of voters.

(*iv*) A $(3,2)$ game is complete *if* it is D^+-complete, D^--complete and $D^\pm$-complete.

Link between weights and desirability relations

Given two arbitrary players p and r in a weighted $(3,2)$ game, for any weight function representing it, we have:

(i) $w^+(p) \geq w^+(r) \Rightarrow p \ \succsim_{D^+} \ r$.

(ii) $-w^-(p) \geq -w^-(r) \Rightarrow p \ \succsim_{D^-} \ r$.

(iii) $w^+(p) - w^-(p) \geq w^+(r) - w^-(r) \Rightarrow p \ \succsim_{D^\pm} \ r$.

Corollary

Every weighted $(3,2)$ game is a complete $(3,2)$ simple game for which the three relations $\succsim_{D^+}$, $\succsim_{D^-}$ and $\succsim_{D^\pm}$ are transitive.

Corollary

A strongly weighted $(3,2)$ game is an I-complete $(3,2)$ game.

Inclusion of types of VRA

I-complete game	$\subset$	complete game
$\cup$		$\cup$
strongly weighted game	$\subset$	weighted game

Inclusion of types of VRA

$$
\begin{array}{ccc}
I\text{-complete game} & \subset & \text{complete game} \\
\cup & & \cup \\
\text{strongly weighted game} & \subset & \text{weighted game}
\end{array}
$$

$$I\text{-complete game} \not\subseteq \text{weighted game}$$

$$\text{weighted game} \not\subseteq I\text{-complete game}$$

A small classification

Every $(3,2)$ simple game with:

- $n = 2$ is I-complete and moreover strongly weighted.
- $n = 3$ is complete. (*)

- (*) Although some of them are neither weighted nor I-complete.
- If $n > 3$ one easily may find examples of (3,2) simple games non-being complete.

Banzhaf's (3,2) extension, $(Bz_p[W])$. Felsenthal and Machover introduced it in 1997. They consider the raw index as:

$$\eta_p[W] = |\{S : S \in W,\ S_{\downarrow p} \notin W\}|$$

The number $\eta_p[W]$ counts the number of winning tripartitions in which p is positively decisive descending one single level of approval.

$$|\{S : p \in S_2,\ S \in W,\ S_{\downarrow p} \notin W\}| = | \{S : p \in S_1,\ S_{\downarrow p} \in W,\ S_{\downarrow\downarrow p} \notin W\} |$$

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$Bz_p[W]$ is a numerical measure of decisiveness.

$$Bz_p[W] = \frac{\eta_p[W]}{\text{total number of tripartitions with } p \in S_1} = \frac{\eta_p[W]}{3^{n-1}}$$

We extend to weighted (3,2) games the idea by Brams and Affuso, which consists of using generating functions to obtain $\eta[W]$ as the sum of some of its coefficients.

We build the *global generating function* $f(x)$ and the *individual generating functions* $f_i(x)$ in the following way:

$$f(x) = \prod_{p \in N} (x^{w^N(p)} + 1 + x^{w^Y(p)}) = \sum_{j=\bar{W}^N}^{\bar{W}^Y} \alpha_j x^j$$

$$f_i(x) = \prod_{p \neq i} (x^{w^N(p)} + 1 + x^{w^Y(p)}) = \sum_{j=\bar{W}^N - w^N(i)}^{\bar{W}^Y - w^Y(i)} \alpha_j x^j$$

where $\bar{W}^Y = \sum_{p \in N} w^Y(p)$, $\bar{W}^N = \sum_{p \in N} w^N(p)$.

$$W^m = \{(12, \emptyset, 3), (1, 23, \emptyset), (23, 1, \emptyset)\} \equiv [2\,;\ (2, -1), (1, -2), (1, -1)]$$

$$f_1(x) = (1 + x + x^{-2})(1 + x + x^{-1}) = x^{-3} + x^{-2} + 2x^{-1} + 2 + 2x + x^2$$
$$f_2(x) = (1 + x^2 + x^{-1})(1 + x + x^{-1}) = x^{-2} + 2x^{-1} + 2 + 2x + x^2 + x^3$$
$$f_3(x) = (1 + x^2 + x^{-1})(1 + x + x^{-2}) = x^{-3} + x^{-2} + x^{-1} + 3 + x + x^2 + x^3$$

$$\left.\begin{array}{l|l} q - w^Y(1) = 0 & q = 2 \\ q - 1 = 1 & q - w^N(1) - 1 = 2 \end{array}\right\} \rightarrow \eta_1[W] = (\alpha_0 + \alpha_1) + a_2 = 4 + 1 = 5$$

$$\left.\begin{array}{l|l} q - w^Y(2) = 1 & q = 2 \\ q - 1 = 1 & q - w^N(2) - 1 = 3 \end{array}\right\} \rightarrow \eta_2[W] = \alpha_1 + (\alpha_2 + \alpha_3) = 2 + 2 = 4$$

$$\left.\begin{array}{l|l} q - w^Y(3) = 1 & q = 2 \\ q - 1 = 1 & q - w^N(3) - 1 = 2 \end{array}\right\} \rightarrow \eta_3[W] = \alpha_1 + \alpha_2 = 1 + 1 = 2$$

$$Bz[W] = \frac{1}{9}(5, 4, 2)$$

Example

The UNSC voting system (as a simple game)

$$[39; 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$$

The UNSC voting system (as a (3,2) game)

$$[\ 9;\ \underbrace{(1,-6), \ldots, (1,-6)}_{5}, \underbrace{(1,0), \ldots, (1,0)}_{10}\ ]$$

Indices	Simple game		(3,2) game	
	Permanent M.	*Nonpermanent M.*	*Permanent M.*	*Nonpermanent M.*
$Rae(\mathcal{W})$	0, 5259	0, 5026	0, 3409	0, 337
$Bz(\mathcal{W})$	0, 0518	0, 0051	0, 0227	0, 0111
$Col^P(\mathcal{W})$	1	0, 0991	1	0, 3621
$Col^I(\mathcal{W})$	0, 0266	0, 0026	0, 0125	0, 0075
$KB(\mathcal{W})$	1	0, 5495	0, 738	0, 5747

Proportion of power for simple games $(Bz \simeq 10 : 1)$
Proportion of power for (3,2) games $(Bz \simeq 2 : 1)$

A different notion of power?

In our notation we have $\eta^+(W) + \eta^-(W) = \eta^\pm(W)$ while both $\eta^+(W)$ and $\eta^-(W)$ are independent.

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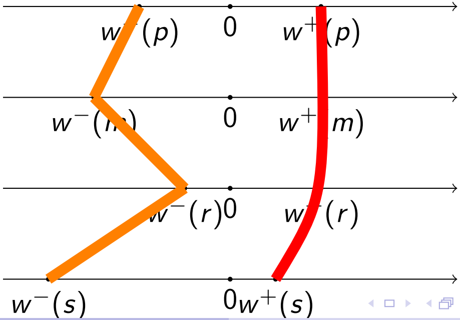
We do think that Banzhaf power in the (3, 2) context should be conceived as 2-component power rather than just as a single numerical measure.



Decomposition in 2-component power for the UNSC:

$$Bz_p^{YA}[V] = 0.0146 \quad Bz_p^{AN}[V] = 0.0080 \quad Bz_p^{YN}[V] = 0.02265$$
$$Bz_p^{YA}[V] = 0.0111 \quad Bz_p^{AN}[V] = 0 \quad Bz_p^{YN}[V] = 0.01111$$

THANKS FOR YOUR ATTENTION