Power in voting games: axiomatic and probabilistic approaches

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Outline

Notion(s) of power

What do you want to measure?

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Depending on your answer we should look for a suitable tool.

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An open definition of "power"...

Roughly speaking...

a power index is a numerical measure that estimates the a priori . . . of each voter in a simple game.

What could the . . . be?

What do you want to measure?

Several options:

- Decisiveness as a payoff:

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- Success: expectation to achieve the desired result
- Inclusiveness: expectation to be part in the winning side
- Luckiness, etc.

Measuring decisiveness as a payoff: P-power

- The property of efficiency ($\sum_{i \in N} \psi_i = 1$) for P-power seems inescapable. Thus, it is a requirement rather than an axiom.

Axiomatic approach: Properties for a power index $\psi : \mathcal{S}_N \to \mathbb{R}^n$:

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Axiomatic approach: Properties for a power index $\psi : \mathcal{S}_N \to \mathbb{R}^n$:

- Efficiency,
- If i is a null voter then $\psi_i[W] = 0$,
- If i and j are equivalent voters then $\psi_i[W] = \psi_j[W]$
- If W and W' are two games, consider $W \vee W'$ and $W \wedge W'$. Then

$$\psi[W] + \psi[W'] = \psi[W \vee W'] + \psi[W \wedge W']$$

How many indices exist with these properties?

...only one!

the Shapley–Shubik index which is the restriction to simple games of the Shapley value for cooperative games.
Formulation for simple games, W_a are the winning coalitions

$$SS_a[W] = \sum_{\substack{S: S \notin W, \\ S \cup \{a\} \in W}} \frac{s!(n-s-1)!}{n!} \qquad (|S| = s)$$

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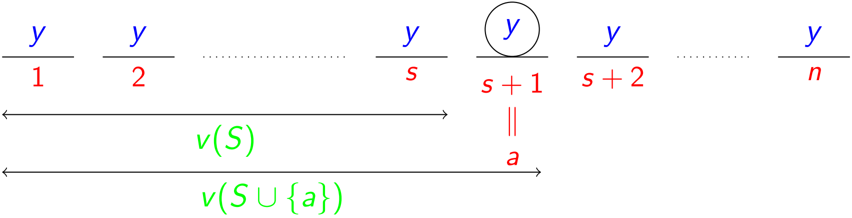
Probabilistic approach: derivation of the model from a bargaining model

Shapley value bargaining model, 1953



- Comment: Rationality of players + superadditivity stimulate the formation of N and to divide revenues or costs according to the Shapley value.

- Question: Is the probabilistic model provided a convincing argument for cooperative games?

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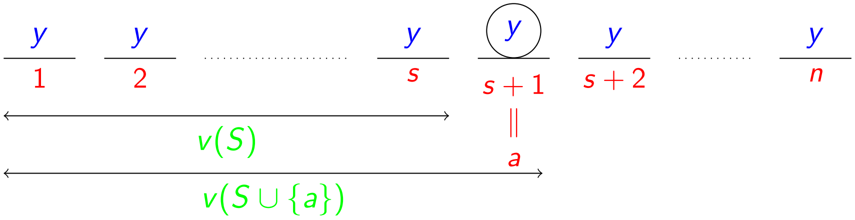

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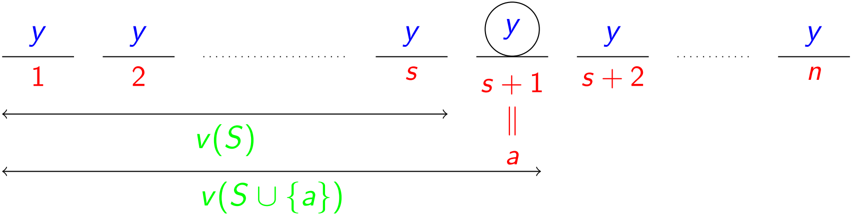

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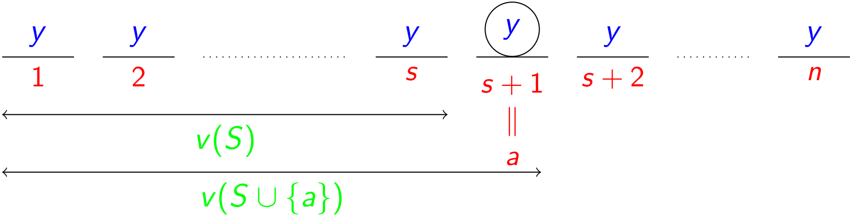

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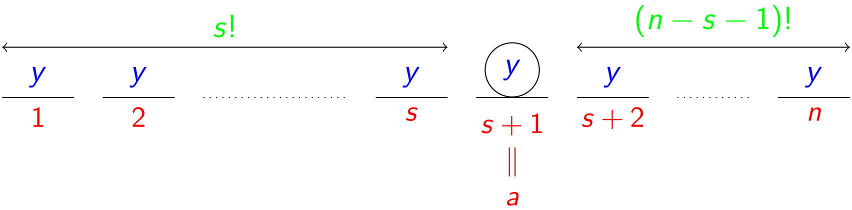
- Question: Is the probabilistic model provided a convincing argument for cooperative games? yes
  and for the "restriction" to simple games? ...maybe "not"

Standard formula for both S-value and SS-index



Well-known formula by taking common factors:

The Shapley value ϕ is given by

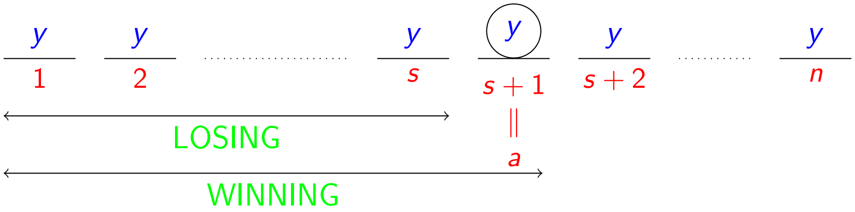
$$\phi_a(v) = \sum_{S \in 2^{N \setminus \{a\}}} \frac{s!(n-s-1)!}{n!} [v(S \cup \{a\}) - v(S)]$$

for any $a \in N$, where $s = |S|$.

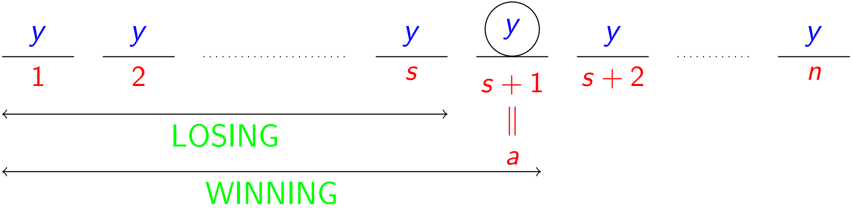
The S&S bargaining model for the S&S index, 1954

1. Assume that all orderings of voters are equally probable.
2. Assume that everybody votes "yes" in his/her turn.
3. A player is "pivotal" if the coalition of his/her predecessors in the queue is losing and his/her addition to it does the new coalition winning.

- The S&S index is the probability of being pivotal under de above scheme, or equivalently
- it is the expected value of the marginal contributions under this scheme.



Question: Is this the most natural probabilistic scheme for the index?



Question: Is this the most natural probabilistic scheme for the index? ...Should it be possible for a voter to vote "no"?

Example Consider the 1958 EU voting system:

$$[12; 4, 4, 4, 2, 2, 1] \equiv [6; 2, 2, 2, 1, 1, 0]$$

$$B\ B\ \underline{B}\ S\ S$$

(we ignore the null voter because receives 0) a big country is 4 times pivotal in the third position

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a big country is 24 times pivotal in the fourth position.

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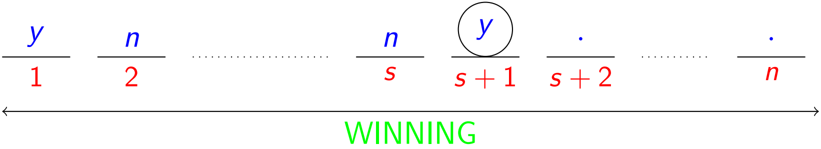
a big country is 24 times pivotal in the fourth position. Thus the power of a big country is $\frac{28}{120} = \frac{7}{30}$, and the power of a small country is: $\frac{1}{2}\left(1 - 3\frac{7}{30}\right) = \frac{3}{20}$.

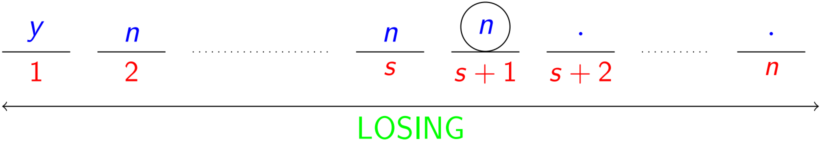
Probabilistic approach

- The Shapley-Shubik index relies for its justification on the axiomatic derivation of the Shapley value, not on any model of voting protocol, bargaining or coalition formation.
- In particular the queue formation procedure of voting is merely a heuristic device for calculating the values of the SS.
- It is not intended as a justification of the Shapley-Shubik index, and is certainly not to be taken seriously as a description of how voting actually takes place.

Alternative bargaining model:

A voter can vote either "in favor" or "against" the proposal submitted to vote.





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- the Johnston index
- the Deegan-Packel index
- the Holler index

they do not come from the cooperative context

- the nucleolus

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Reasonable requirements for P-power

Minimum requirements:

- E Efficiency
- N+ET Null and equal treatment properties
- S Sensitivity
  - Weak sensitivity: if $v(S \cup \{i\}) \geq v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$, then $\psi_i(v) \geq \psi_j(v)$
  - Strong sensitivity: if, moreover, $v(S \cup \{i\}) > v(S \cup \{j\})$ for some $S \subseteq N \setminus \{i, j\}$, then $\psi_i(v) > \psi_j(v)$.

Only ... SS, Bz, Jh fulfill them.

Some other reasonable requirements for P-power

D Let u and w two simple games, and $\mathcal{C}_i(u) \subset \mathcal{C}_i(w)$, then $\psi_i(u) \leq \psi_i(w)$.

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- AV Let u and w be derived from u by adding a veto player i, then for all $j, k \in N$ the power of the players in N should remain proportional

$$\psi_j(u)/\psi_k(u) = \psi_j(w)/\psi_k(w)$$

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No known index satisfies all these axioms: E+N+ET+S+D+AV (conjecture)

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No known index satisfies all these axioms: E+N+ET+S+D+AV (conjecture)
No index satisfies: E+N+ET+T+AV (trivial impossibility theorem)

Measuring...decisiveness as influence

The Banzhaf index is simply the probability of being crucial in the game.

$$Bz_a[W] = \frac{|\mathcal{C}_a[W]|}{2^{n-1}} = \text{Probability for } a \text{ of being crucial}$$

where $\mathcal{C}_a[W] = \{S \subseteq N \ : \ S \in W,\, S \setminus \{a\} \notin W\}$.

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where $\mathcal{C}_a[W] = \{S \subseteq N \ : \ S \in W, \, S \setminus \{a\} \notin W\}$.

- Only a winning coalition will be formed.
- All winning coalitions have equal probability of being formed.
- All crucial voters receive equal shares.

Efficiency is <u>not</u> a requirement.

Consider the 1958 EU voting system:

$$[12; 4, 4, 4, 2, 2, 1] \equiv [6; 2, 2, 2, 1, 1, 0])$$

$$\mathcal{C}_B[W] = \{3B, 3B + S, 2B + 2S\}$$

models that represent 1, 2 and 2 coalitions respectively.

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$$\mathcal{C}_S[W] = \{2B + 2S\}$$

model that represents 3 coalitions.

$$Bz[W] = \left(\frac{5}{16}, \frac{5}{16}, \frac{5}{16}, \frac{3}{16}, \frac{3}{16}, 0\right)$$

To emulate the SS index, some scholars provide axiomatizations for the Banzhaf (value) index.

- Owen (1978)
- Lehrer (1988) (Banzhaf value)
- Feltkamp (1995)
- Barua et al. (2005)

Measuring success, luckiness and inclusiveness

Success:

$$Rae_i[W] = \frac{|S \,:\, i \in S \in W|}{2^n} + \frac{|S \,:\, i \notin S \notin W|}{2^n}$$

Measuring success, luckiness and inclusiveness

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Inclusiveness:

$$KB_i[W] = \frac{|W_i|}{|W|} \qquad \text{where } W_i = \{S \in W : i \in S\}$$

$$W^m = \{\{1,2\},\{1,3\},\{2,3\}\}$$

$$Rae_1[W] \to \{1,2\},\{1,3\},\{1,2,3\},\{2\},\{3\},\{\emptyset\}$$

$$Luc_1[W] \to \{\emptyset\},\{1,2,3\}$$

$$Rae_1[W] = \frac{6}{8} = \frac{3}{4}, \quad Luc_1[W] = \frac{2}{8} = \frac{1}{4}, \quad Bz_1[W] = \frac{1}{2}$$

Observe...

$$W^m = \{\{1,2\},\{1,3\},\{2,3\}\}$$

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Observe...

$$Rae_1[W] = Bz_1[W] + Luc_1[W]$$

but also

$$Rae_1[W] = \frac{1}{2} + \frac{1}{2} Bz_1[W].$$

Barry's equation and Dubey and Shapley's equation:

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Success = Decisiveness + Luckiness

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linear relationship between success and decisiveness

Model: $\boxed{((N, W), p)}$

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Assume that a probability distribution over vote configurations p (exogenous information) enters as a second input besides the simple game W. Of course,

$$0 \leq p(S) \leq 1 \text{ for all } S \subseteq N, \quad \text{and} \quad \sum_{S \subseteq N} p(S) = 1.$$

In a voting situation thus described by the pair (W, p) the ease of passing proposals or probability of acceptance is given by

$$\alpha(W, p) = Prob\ (\text{acceptance}) = \sum_{S : S \in W} p(S)$$

Success and Decisiveness in the model $((N, W), p))$

$$\Omega_i(W,p) = Prob\ (i \text{ is successful}) = \sum_{S: i \in S \in W} p(S) + \sum_{S: i \notin S \notin W} p(S).$$

$$\Phi_i(W,p) = Prob\ (i \text{ is decisive}) = \sum_{\substack{S: i \in S \in W \\ S \setminus \{i\} \notin W}} p(S) + \sum_{\substack{S: i \notin S \notin W \\ S \cup \{i\} \in W}} p(S).$$

$$\Lambda_i(W,p) = Prob\ (i \text{ is lucky}) = \sum_{\substack{S: i \in S \\ S \setminus \{i\} \in W}} p(S) + \sum_{\substack{S: i \notin S \\ S \cup \{i\} \notin W}} p(S)$$

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Barry's equation is still true: success = decisiveness + luckiness

$$\Omega_i(W,p) = \Phi_i(W,p) + \Lambda_i(W,p)$$

Dubey and Shapley's equation:

$$\Omega_i(W,p) \neq \frac{1}{2} + \frac{1}{2}\Phi_i(W,p)$$

does not extend in the more general context $((N, W), p)$

- Only for $\mathbf{p} = \left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$ the equality

  $$\Omega_i(W,p) = \frac{1}{2} + \frac{1}{2}\Phi_i(W,p)$$

  holds.
- For $\mathbf{p} = (p, \ldots, p)$ the two indices rank voters in the same way, but
- For $\mathbf{p} \neq (p, \ldots, p)$ is is always possible to find W such that the rankings do not coincide.

To be or not to be?

What is more important, to be decisive or to be successful?

Questions?

THANKS FOR YOUR ATTENTION