Basic (mathematical) knowledge on the structure of simple games

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Outline

Definition

A simple game is a pair (N, W) where $N = \{1, 2, \ldots, n\}$ is the set of voters or players and $W \subseteq 2^N$ such that:

- $N \in W$,
- $\emptyset \notin W$,
- $S \in W$ and $S \subset T$ implies $T \in W$ (monotonicity).

The set W is called the set of winning coalitions.

Example

Set $N = \{1, 2, 3\}$ and $W = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$,

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Note... a simple game can also be described by the set of minimal winning coalitions, W^m (instead of W):
$S \in W^m$ if $S \in W$ and $T \subset S$ implies $T \notin W$.

$$W^m = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$$

Definition

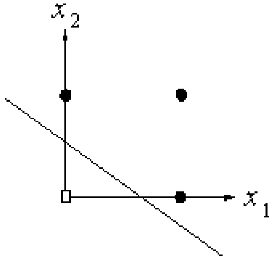
A simple game W in N is a weighted game if and only if there exist nonnegative weights $w_1, w_2, \ldots, w_n$ allocated to the players and a quota $q \in (0, \sum_{i \in N} w_i]$ such that

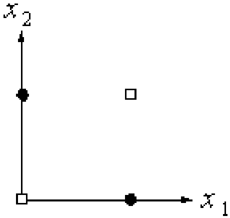
$$S \in W \quad \Leftrightarrow \quad \sum_{i \in S} w_i \geq q$$

We then write $W \equiv [q; w_1, w_2, \ldots, w_n]$

Two simple games with set of players $N = \{1, 2\}$
the first is monotonic the second isn't.

$W = \{\{1\}, \{2\}, \{1, 2\}\}$ $\qquad$ $W = \{\{1\}, \{2\}\}$





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$L^M = \{\{1,3\},\{1,4\},\{2,3\},\{2,4\}\}$

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If it was weighted, there would be a weighted representation $[q; w_1, w_2, w_3, w_4]$ for W such that:

$$\begin{array}{rcl} w_1 + w_2 & > & w_1 + w_3 \\ w_3 + w_4 & > & w_2 + w_4 \\ \hline w_1 + w_2 + w_3 + w_4 & > & w_1 + w_2 + w_3 + w_4 \end{array}$$

a contradiction

Example

$[51; 49, 49, 2]$, $[51; 49, 48, 3]$, $[65; 34, 33, 33]$, $[2; 1, 1, 1]$ are weighted representations for $W^m = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$.

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Example (Former EU in 1958)

$[12; 4, 4, 4, 2, 2, 1] \equiv [6; 2, 2, 2, 1, 1, 0]$
$W^m = \{\{G, F, I\}, \{G, F, N, B\}, \{G, I, N, B\}, \{F, I, N, B\}\} \equiv \{3B, 2B + 2M\}$.
$\overline{n} = (3, 2, 1)$ and models of coalitions $(3, 0, 0)$ and $(2, 2, 0)$ representing 1 and 3 coalitions resp.

Integer representations

- If $[q; w_1, \ldots, w_n]$ is a representation for a weighted game W, then $[c \cdot q; c \cdot w_1, \ldots, c \cdot w_n]$ with $c > 0$ is also a representation for W.

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- Each weighted game admits a representation in integers. (Proof in next slide)
- Consider the set of integer representations for a weighted game. Does it have a unique element?

As N is a finite set and $w(S) < q$ for $S \notin W$, there exist numbers $\delta, \epsilon > 0$ such that

$$w(S) + \epsilon < q - \delta \qquad \forall\ S \notin W$$

Let q' be a rational number such that $q - \delta < q' < q$ and define for each $i \in N$ a rational number w_i' such that $w_i < w_i' < w_i + \epsilon/n$. Then $0 < q' < q \leq w(N) < w'(N)$, thus $[q'; w']$ is a representation for a weighted game (N, W'). We need to check that $W = W'$.

- If $S \in W$ then $w'(S) > w(S) \geq q > q'$, therefore $S \in W'$.
- If $S \notin W$ then

$$w'(S) \leq w(S) + \frac{\epsilon s}{n} < w(S) + \epsilon < q - \delta < q'.$$

  Thus, $S \notin W'$.

let c be the least common multiple of the denominators in q' and $w_1', \ldots, w_n'$. We get the new representation $[c \cdot q'; c \cdot w_1', \ldots, c \cdot w_n']$ for W in integers.

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- For $n = 8$ there exist 154 weighted games with 2 minimal-sum representations. For all these games the affected players are substitutes.
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- For $n = 9$ there are examples with 3 minimal-sum representations or representations in which the affected players are not substitutes:
    - $[54; 14, 11, 7, 5, 5, 5, 3, 2, 2]$ and $[54; 14, 12, 6, 5, 5, 5, 3, 2, 2]$

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Not always exists a minimum integer representation in integers.

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- Almost all simple games have small dimension.
- The system to amend the Canadian Constitution has dimension 2.
  - $[51; 32, 31, 10, 9, 5, 4, 4, 3, 3, 1] \cap [7; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$
- The EU voting system approved in Nice in 2001 has dimension 3.

- The dimension is upper bounded by $\mid L^M \mid$, the number of maximal losing coalitions. (Proof in next slide)

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- There are sequences of games with m voters for which the dimension grows exponentially.

Proof for a particular game. Let $n = 4$ and $W^m = \{\{1,2\},\{3,4\}\}$. Then, $L^M = \{\{1,3\},\{1,4\},\{2,3\},\{2,4\}\}$. For each maximal losing coalition consider a weighted game for which it loses: $[1;0,1,0,1]$ $[1;0,1,1,0]$ $[1;1,0,0,1]$ $[1;1,0,1,0]$.

$$W = [1;0,1,0,1] \cap [1;0,1,1,0] \cap [1;1,0,0,1] \cap [1;1,0,1,0]$$

Thus, the dimension of this game is at most 4.

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<u>Exercise:</u> We know: $2 \leq dim(W) \leq 4$.
Which is the dimension of this game?

Proof for a particular game. Let $n = 4$ and $W^m = \{\{1,2\},\{3,4\}\}$. Then, $L^M = \{\{1,3\},\{1,4\},\{2,3\},\{2,4\}\}$. For each maximal losing coalition consider a weighted game for which it loses:
$[1;0,1,0,1]$ $[1;0,1,1,0]$ $[1;1,0,0,1]$ $[1;1,0,1,0]$.

$$W = [1;0,1,0,1] \cap [1;0,1,1,0] \cap [1;1,0,0,1] \cap [1;1,0,1,0]$$

Thus, the dimension of this game is at most 4.

Exercise: We know: $2 \leq dim(W) \leq 4$.
Which is the dimension of this game?

Exercise: Let $N = \{1,2,\ldots,2n-1,2n\}$ and let W such that

$$S \in W \qquad \Longleftrightarrow \qquad S \cap \{2i-1,2i\} \neq \emptyset \quad \text{for } i = 1,2,\ldots,n$$

Prove that W has dimension n.

Definition

Desirability relation. Let $i, j \in N$.

$$i \succsim_D j \quad \text{iff} \quad S \cup \{j\} \in W \Rightarrow S \cup \{i\} \in W \quad \text{for all} \quad S \subseteq N \setminus \{i,j\}.$$

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$i \succsim_D j$ iff $S \cup \{j\} \in W \Rightarrow S \cup \{i\} \in W$ for all $S \subseteq N \backslash \{i, j\}$.

The game is complete if $\succsim_D$ is total.

The desirability relation $\succsim_D$ is a preordering on N (i.e., $\succsim_D$ is a reflexive and transitive relation).

We write

- $i \succ_D j$ if $i \succsim_D j$ but $j \not\succsim_D i$, and
- $i \approx_D j$ stands for $i \succsim_D j$ and $j \succsim_D i$.

Example (System to Amend the Canadian Constitution)

$$[51; 32, 31, 10, 9, 5, 4, 4, 3, 3, 1] \cap [7; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$$

It is a complete game, since Big province $\succ_D$ Small province

Observation

x: weight for Ontario and Quebec (Big provinces)
y: weight for the other eight provinces (Small provinces)
This game is not weighted!

$$\begin{array}{rcl} x + 6y & > & 2x + 4y \\ x + 6y & > & 8y \\ \hline 2x + 12y & > & 2x + 12y \quad \text{(a contradiction!)} \end{array}$$

Example

In the former EU $[12; 4, 4, 4, 2, 2, 1]$:

$$I \approx_D G \approx_D F \succ_D N \approx_D B \succ_D L$$

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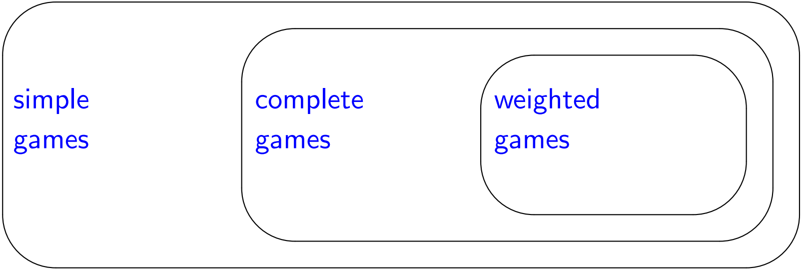
$$I \approx_D G \approx_D F \succ_D N \approx_D B \succ_D L$$

Note that... $w_i \geq w_j$ implies $i \succsim_D j$. Thus, every weighted game is complete.

Inclusion of simple games

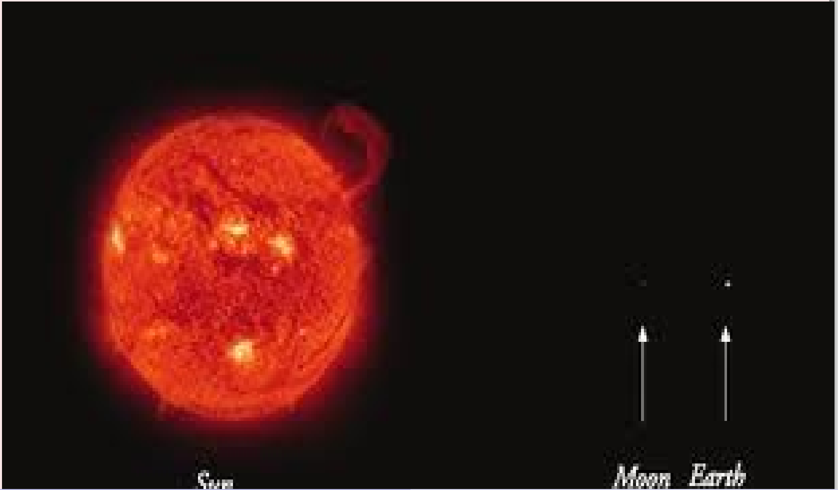


Known enumerations

n	1	2	3	4	5	6	7	8	9
CGs	1	3	8	25	117	1171	44313	16175188	284432730174
WGs	1	3	8	25	117	1111	29373	2730164	989913344

Table: Number of complete (CGs) and weighted games (WGs) with n voters.

An approximate comparison for 8 voters:



Example

There exist (only for $n > 5$) complete games non-being weighted games.

$$\begin{aligned} \mathcal{W}^m \quad = \quad & \{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\}, \\ & \{1,3,4\},\{1,3,5\},\{1,3,6\},\{1,4,5\},\{1,4,6\},\{1,5,6\} \\ & \{2,3,4\},\{2,3,5\},\{2,3,6\},\{2,4,5\},\{2,4,6\},\{2,5,6\}\} \end{aligned}$$

or, alternatively, as the intersection of weighted games

$$[3;1,1,1,1,1,1] \cap [5;3,3,1,1,1,1]$$

Example

Any system of weights should satisfy

$$w_2 + w_5 + w_6 > w_1 + w_2, \qquad w_1 + w_3 + w_4 > w_3 + w_4 + w_5 + w_6$$

and hence

$$w_5 + w_6 > w_1, \qquad w_1 > w_5 + w_6$$

a contradiction. Nevertheless, the game is complete, and the equi-desirable classes are:

$$N_1 = \{1, 2\} > N_2 = \{3, 4, 5, 6\}.$$

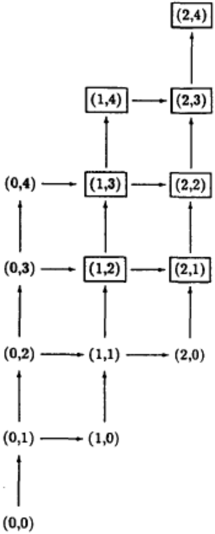
Example

Are all real–world binary voting systems modeled as complete games? Almost all of them...

- weighted games: $[q_1; w_1, ..., w_n]$,
- voting by count and account $[q_1; w_1, ..., w_n] \cap [q_2; 1, ..., 1]$,
- triple or fourfold... weighted ordered intersections. E.g., the European Economic Union (after 25 and 27 member enlargements) $[q_1; w_1, ..., w_n] \cap [q_2; v_1, ..., v_n] \cap [q_3; u_1, ..., u_n]$ (with weights in non–increasing order)

Few exceptions...

- The United States Federal System voting system is not complete. E.g., a senator and a member of the representative chamber are not comparable by $\succsim_D$.



The lattice associated with previous Example.

Let $(N, \mathcal{W})$ be a complete simple game.
Let $N_1 > N_2 > \cdots > N_t$ be the linear ordering of the equivalence classes, and let $\overline{n} = (n_1, n_2, \ldots, n_t)$ be the vector defined by their cardinalities.

Finally, let

$$\mathcal{M} = \begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1t} \\ m_{21} & m_{22} & \cdots & m_{2t} \\ \ldots & \ldots & \ldots & \ldots \\ m_{r1} & m_{r2} & \cdots & m_{rt} \end{pmatrix}$$

be the matrix whose rows $\overline{m}_p = (m_{p1}, m_{p2}, \ldots, m_{pt})$ are the models of shift-minimal winning coalitions.

Theorem

The vector $\overline{n}$ and the matrix $\mathcal{M}$ associated with a complete simple game $(N, \mathcal{W})$ satisfy the following properties:

1. $n_k > 0$ *for* $k = 1, 2, \ldots, t$;
2. $0 \leq \overline{m}_p \leq \overline{n}$ *for* $p = 1, 2, \ldots, r$;
3. $\overline{m}_p$ *and* $\overline{m}_q$ *are not comparable by partial sums if* $p \neq q$; *and*
4. *if* $t = 1$, *then* $m_{11} > 0$; *if* $t > 1$, *then for every* $k < t$ *there exists some* p *such that* $m_{pk} > 0$, $m_{p(k+1)} < n_{k+1}$.

Theorem

(a) *Two complete simple games* $(N, \mathcal{W})$ *and* $(N', \mathcal{W}')$ *are isomorphic if and only if* $\overline{n} = \overline{n}'$ *and* $\mathcal{M} = \mathcal{M}'$*.*

(b) *Given a vector* $\overline{n}$ *and a matrix* $\mathcal{M}$ *satisfying the conditions in previous Theorem, there exists a complete simple game* $(N, \mathcal{W})$ *the characteristic invariants of which are* $\overline{n}$ *and* $\mathcal{M}$*.*

Example

$$\overline{n} = (3, 4, 2), \qquad \mathcal{M} = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 3 & 1 \\ 0 & 4 & 2 \end{pmatrix}$$

$$N_1 = \{1, 2, 3\} > N_2 = \{4, 5, 6, 7\} > N_3 = \{8, 9\}$$

The shift-minimal winning coalitions are given by the models:

$$(2, 2, 0), \qquad (1, 3, 1) \quad \text{and} \quad (0, 4, 2),$$

and the remaining minimal winning coalitions by the models:

$$(3, 1, 0) \quad \text{and} \quad (1, 4, 0).$$

Model	Number	Minimal winning coalitions
$(2,2,0)$	18	$\{1,2,4,5\},\{1,2,4,6\},\{1,2,4,7\},\ldots$
$(2,3,1)$	24	$\{1,4,5,6,8\},\{1,4,5,6,9\},\{1,4,5,7,8\},\ldots$
$(0,4,2)$	1	$\{4,5,6,7,8,9\}$
$(3,1,0)$	4	$\{1,2,3,4\},\{1,2,3,5\},\{1,2,3,6\},\{1,2,3,7\}$
$(1,4,0)$	3	$\{1,4,5,6,7\},\{2,4,5,6,7\},\{3,4,5,6,7\}$

Example (Is the game weighted?)

$$\overline{n} = (3,4,2), \qquad \mathcal{M} = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 3 & 1 \\ 0 & 4 & 2 \end{pmatrix}$$

$$\overline{m}_1 = (2,2,0), \qquad \overline{m}_2 = (1,3,1), \quad \text{and} \quad \overline{m}_3 = (0,4,2),$$

Models of shift-maximal models of losing coalitions.

$$\overline{\alpha}_1 = (3,0,1), \qquad \overline{\alpha}_2 = (1,3,0) \quad \text{and} \quad \overline{\alpha}_3 = (0,4,1),$$

Inequality system with unknowns $w = (w_1, w_2, w_3)$

$$(\overline{m}_p - \overline{\alpha}_q) \cdot w > 0 \quad \text{for } p = 1,2,3, \; q = 1,2,3.$$

By solving this system we find a solution:

$$w_1 = 5, \; w_2 = 4 \text{ and } w_3 = 1,$$

and hence

$$[18; 5, 5, 5, 4, 4, 4, 4, 1, 1]$$

Simplification

Let $n > 2$ and let $(N; \mathcal{W})$ be a complete simple game such that $\overline{n} = (1, 1, \ldots, 1)$. Then the equalities

$$\mathcal{W}^{sm} = \mathcal{W}^{m} \text{ and } \mathcal{L}^{sM} = \mathcal{L}^{M}$$

are incompatible.

Some enumerations of complete simple games

- Number of complete games of n voters and with $r = 1$:

$$2^n - 1$$

- Number of complete games of n voters and with $t = 2$:

$$F(n+6) - (n^2 + 4n + 8)$$

where $F(n)$ are the Fibonacci numbers: $F(0) = 0$, $F(1) = 1$, and $F(n) = F(n-1) + F(n-2)$ for all $n > 1$.

Questions?

THANKS FOR YOUR ATTENTION