Varieties of Monotonicity among Voting Systems

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Main points

- the concept of monotonicity
- majority rule and monotonicity
- Maskin monotonicity is uncommon among voting systems
- the brief and curious history of the no-show paradox
- the strong version of the no-show paradox
- P-BOT paradox
- varieties of Borda elimination

Definition

Monotonicity says that additional support, *ceteris paribus*, never turns winners into non-winners. The ceteris paribus clause is very important here.

Definition

Let $R = (R_1, \ldots, R_n)$ be a n-person profile of complete and transitive preference relations over alternative set A. Denote by P_i the asymmetric part of the relation. Assume that $F(A, R) = x \in A$, i.e. rule F when applied to profile R over alternative set A results in x being elected. Consider now any $R' = (R'_1, \ldots, R'_n)$ such that for all $y, z \in A (x \neq y, x \neq z)$ and for all $i \in N : yR_i z$ if and only if $yR_i z$, while for at least one $i \in N$ and for at least one $w \in A : wP_i x$, but $xP'_i w$ and for no $i \in N$ and for no $v \in A : xR_i v$ with $vR'_i x$. Now, F is monotonic if under the preceding conditions $F(A, R') = x$.

Nonmonotonicity of Hare and pl runoff

22 voters	21 voters	20 voters
A	B	C
B	C	A
C	A	B

Table : Nonmonotonicity of pl runoff and Hare

Pl runoff or Hare $\rightarrow A$.
With 2 voters in the middle lifting A first - *ceteris paribus* - C wins. So additional support may turn winners into non-winners.

Nonmonotonicity of Dodgson's rule

42 voters	*26 voters*	*21 voters*	*11 voters*
b	a	e	e
a	e	d	a
c	c	b	b
d	b	a	d
e	d	c	c

Table : Nonmonotonicity of Dodgson

Alternative a wins with only 14 binary preference reversals to become the Condorcet winner. Now, suppose that the 11 right-most voters increase the support of a by ranking it first, *ceteris paribus*. After the change, b is immediately below e in the 11-voter ranking and b needs only 9 binary preference changes to become the Condorcet winner, while a still needs 14. Therefore, the new winner is b.

Monotonicity, domination and Condorcet

So, there are Condorcet extensions and elimation systems that are nonmonotonic, one may ask whether all Condorcet extensions are nonmonotonic. The answer is: no. Copeland's system is monotonic as is min-max.

What about positional systems, e.g. plurality and Borda count? They are both monotonic as are all systems based on non-increasing scoring for ranks from best to worst.

Definition

Let x_i denote the number of voters assigning alternative x to ith rank $(i = 1, \ldots, m)$. The x positionally dominates y or x D y, iff $\sum_{i=1}^{k} x_i > \sum_{i=1}^{k} y_i$, for $k = 1, \ldots, m-1$.

Definition

xMy iff there are more voters ranking x higher than y than there are voters ranking y higher than x.

Theorem

Fishburn 1982. Consider $A = \{a, b, c\}$. Suppose that the rule F being applied has the following property under any profile: if aDc, bDc and aMb, then $F(A) = \{a\}$. Then F is nonmonotonic.

Smith's result

Definition

Point runoff system is any aggregation system where the ranks are determined as follows:

1. a monotonic point system is applied to all N candidates; a second monotonic point system is applied to the highest ranking $N - K_1$ candidates using the voters' rankings on those candidates only; after some number of stages afinal monotonic point system is applied to the remaining K_r candidates to place them in the top K_r ranks,
2. after the K_r places have been determined by the above, a similar process in applied to the remaining candidates placing some of them at the top of the remaining ranks.

Smith's result, cont'd

Theorem

Smith 1973. No point runoff system involving two or more stages and non-trivial point systems is monotonic. More precisely, if such a system determines first place first, then a change of votes in a candidate's favor can remove him from the first place . . ..

N.B. Nanson's method is NOT a point runoff system.

Defining the concept

Suppose that alternative $x \in A$ is selected when the profile is $R = R_1, \ldots, R_n$. Then form a profile $S = S_1, \ldots, S_n$ over A so that for any alternative $y \in A$, the position of x with respect to y is at least as high in S as in R and for some individuals $j \in N$, the position of x *vis-à-vis* y is strictly higher in S. Maskin monotonicity requires that x be selected in S as well.
N.B. There is no *ceteris paribus* clause in this definition.
It turns out that all main voting procedures fail on Maskin monotonicity.

Plurality fails on Maskin monotonicity

2 voters	*1 voter*	*1 voter*	*1 voter*	*2*	*1*	*1*	*1*
A	B	C	D	A	B	**B**	**B**
B	C	B	C	B	**A**	**A**	**A**
C	A	A	B	C	**C**	**C**	**D**
D	D	D	A	D	D	D	**C**

Table : Plurality and Maskin monotonicity

On the left sub-profile A wins. On the right, B wins. Yet A is equally high or higher on every voters ranking on the right as on the left.

Borda fails on Maskin monotonicity

4 voters	*3 voters*	*3 voters*	*4*	*1*	*2*	*3*
A	B	C	A	B	**C**	C
B	C	A	**C**	**A**	**B**	A
C	A	B	**B**	**C**	A	B

Table : Borda count and Maskin monotonicity

On the left A wins. On the right C wins and yet A is as high or higher on every voter's ranking on the right.

On the matter of definition

Fishburn and Brams (1983, p. 207) called No-Show Paradox a phenomenon according to which

> *The addition of identical ballots with candidate x ranked last may change the winner from another candidate to x.*

They then proceed to show that the plurality with runoff procedure is vulnerable to this paradox. Thereafter they present another no-show paradox – which they attribute to a 1910 Report of the Royal Commission Appointed to Enquire into Electoral Systems (HMSO, London), as well as to Meredith (1913). This other no-show paradox is shown by Fishburn and Brams to afflict the STV procedure and is described by them as follows

> *This other paradox says that one of the candidates elected by preferential voting [aka STV] could have ended up a loser if additional people ranked him in first place had actually voted.*

Moulin (1988, pp. 53-54) writes:

> *Following Brams and Fishburn [1983] we call No Show Paradox a situation where a voter is better off not showing up (as this leads to the election of a candidate whom he prefers).*

It seems that the latter (Moulin's) interpretation is now the most common one.
Vulnerability to no show paradox $\Leftrightarrow$ violation of the participation axiom.

Are all nonmonotonic systems vulnerable to so-show?

I.e. is the lower right-hand cell empty?

Table : Monotonicity and vulnerability to no-show paradox

	monotonic systems	nonmonotonic systems
vulnerable	Copeland	alternative vote
invulnerable	Borda count	

Are all...?, cont'd

Yes and no.

Theorem

(Campbell and Kelly). *Nonmonotonicity does not imply the no-show paradox.*

Example. Consider an alternative $x \in X$ and a subset J of active (voting) voters in the set of N of all voters. We assume that there are more than two voters. Given a profile P, define now choice rule g as follows:

$$g(J, P) = x$$

if x is bottom-ranked by all $i \in J$, and otherwise,

$$g(J, P) = y$$

where y is the top-ranked alternative of the smallest element of J not ranking x at the bottom.

Are all...?, cont'd

In other words, given the preferences reported by the voters, the rule elects alternative x if every active voter ranks x lowest. Otherwise, the alternative that is ranked highest by the smallest group of voters is chosen.
This intuitively strange choice rule is, indeed, nonmonotonic as an improvement of the ranking of a winner - provided it is the unanimously bottom-ranked alternative - makes it "very often" non-winning. Yet, g does not exhibit the no-show paradox, since by not voting no group of voters can improve the outcome from what it is if it is voting.

The no-show paradox

35 voters	25 voters	15 voters	25 voters
A	B	B	C
B	C	C	A
C	A	A	B

Table : The weak version and pl runoff (Hare)

This illustrates the latter interpretation of Brams and Fishburn. N.B. with 3 alternatives plurality runoff and STV are equivalent.

The strong version of the paradox: amendment

2 voters	3 voters	2 voters	2 voters
A	B	C	C
B	C	A	B
C	A	B	A

Table : Strong no show and amendment

The amendment agenda: (i) A vs. B, (ii) the winner vs. C, yields B. If the rightmost two voters abstain, the winner is C, their first ranked candidate.

Another example: Copeland

This example is adapted from Richelson (1978, p. 174) who copied it from an example used by Fishburn (1977, p. 483) in a somewhat different context.

2 voters	1 voter	1 voter
E	C	D
D	B	C
A	A	B
B	E	A
C	D	E

Table : Strong no show and Copeland

Copeland, cont'd

	A	B	C	D	E
A	-	2	2	1	2
B	2	-	2	1	2
C	2	2	-	1	2
D	3	3	3	-	1
E	2	2	2	3	-

Table : Pairwise comparison matrix of preceding table

Here candidate D gets 3 points (because it beats three candidates). So according to Copeland's procedure D is the winner. Now suppose that, ceteris paribus, an additional voter with preference ordering $D \succ E \succ A \succ B \succ C$ joins the electorate. As a result E becomes the Condorcet winner and hence is elected according to Copeland's procedure.

P-BOT paradox defined

Felsenthal and Tideman (2013) the P-BOT paradox occurs if:

> *. . . one of the candidates, say candidate c, who has not been elected originally, may be elected if, ceteris paribus, the electorate is increased as a result of additional voters whose bottom-ranked candidate is c join the electorate, and consequently these additional voters are worse off.*

Kemeny is vulnerable to P-BOT

5 voters	3 voters	3 voters
D	A	A
B	D	D
C	C	B
A	B	C

Table : Kemeny and P-BOT

A, the strong Condorcet winner, is at the top of the Kemeny ranking. Now suppose that, ceteris paribus, 4 additional voters whose preference ordering is $B \succ C \succ A \succ D$ join the electorate. The ensuing Kemeny ranking is now: $D \succ B \succ C \succ A$. Hence the lowest ranked candidate of those 4 voters becomes the winner.

Majority judgment is vulnerable to P-BOT

Felsenthal and Machover (2008, p. 329). Suppose that five voters, V1-V5, grade two candidates, x and y, on an ordinal scale ranging between A (lowest) and F (highest), as follows:

voter	V1	V2	V3	V4	V5	median grade
candidate						
x	A	D	E	E	F	E
y	B	C	F	F	F	F

Since the median grade of candidate y (F) is higher than that of candidate x, candidate y is elected according to the MJ procedure. Given that candidate x has not been elected, suppose now that, ceteris paribus, two additional voters, V1* and V1**, join the electorate assigning to candidates x and y the same (or similar) grades as those assigned by voter V1 (i.e., the lowest grade to x and a higher grade to y not exceeding C).

Majority judgment, cont'd

As a result we get:

voter	V1	V1*	V1**	V2	V3	V4	V5	median grade
candidate								
x	A	A	A	D	E	E	F	D
y	B	B	B	C	F	F	F	C

Now x is elected.

Copeland in vulnerable to P-BOT

5 voters	4 voters
B	C
C	D
D	A
A	B

Table : Copeland and P-BOT

	A	B	C	D
A	-	4 (7)	0 (3)	0 (3)
B	5	-	5 (8)	5
C	9	4	-	9
D	9	4 (7)	0 (3)	-

Table : Pairwise comparison

Copeland, cont'd

Here B wins (it is the Condorcet winner).
Suppose now that 3 voters with ranking $A \succ D \succ B \succ C$ join in. Now both C and D become Copeland winners. Thus, the worst-ranked C of the entrants is now included in the choice set showing the vulnerability of Copeland to P-BOT paradox, albeit in somewhat milder form.

Summary

method	vuln. to P-TOP	vuln. to P-BOT
Successive Elimination	Yes	Yes
Bucklin	No	Yes
Majority Judgment	No	Yes
Minimax	No	No
Black	Yes	Yes
Copeland	Yes	Yes
Kemeny	Yes	Yes
Schwartz	Yes	Yes
Young	Yes	No

Table : Summary of vulnerability of some systems to P-TOP and P-BOT

Borda elimination and Nanson' rule

Borda elimination rule (BER) is a version of Borda's rule. According to Borda's rule voters rank-order all the n candidates. Thereafter each candidate i receives n-1 points from each ballot in which i is ranked first, n-2 points from each ballot in which i is ranked second, and so on, and *0* points for each ballot in which i is ranked last. Each candidate's points are summed over all ballots and the candidate with the largest sum of points is elected.

BER-NER, cont'd

The only difference between Borda's rule and BER is that under Borda's rule only a single counting round is conducted and the candidate with the highest number of points is elected, whereas under BER up to *n-1* counting rounds can be conducted in each of which the candidate(s) with the lowest Borda score is (are) eliminated and the remaining candidates' Borda scores are re-calculated. The elimination process continues in this way until only two candidates are left and the one whose Borda score is highest is declared the winner. If the remaining candidates are tied then some tie-breaking rule must be applied to determine the winner.

BER-NER, cont'd

Nanson's elimination rule (NER) was proposed by E.J. Nanson in 1883. Whereas under both BER and NER the Borda scores awarded to the remaining candidates in a given counting round is the same, the elimination process under NER may be considerably shorter than that under BER. This is so because at the end of every counting round all candidates whose Borda scores are *equal to, or lower than, the average Borda score* of all candidates in that counting round are eliminated. As a result of this difference in the elimination process between BER and NER these two procedures may result, *ceteris paribus*, in the election of different candidates when no Condorcet Winner exists.

BER $\neq$ NER (Niou)

1	1	1	1	1
A	C	C	B	B
B	A	A	C	C
C	B	B	A	A

Here NER yields C, while BER gives B.

What about properties of BER and NER?

- both are Condorcet extensions
- both fail on monotonicity – but not necessarily on the same profiles
- there are profiles where BER fails, but NER doesn't and vice versa
- NER never fails in profiles involving at most three voters, but BER may fail

BER, but not NER fails on monotonicity when $k = 3$

P. C. Fishburn

8	5	5	2
A	C	B	C
B	A	C	B
C	B	A	A

The Borda scores are 21, 20, 19. Hence C is eliminated whereupon A wins. Suppose now that the 2-voter group switches from CBA to CAB, i.e. in favor of A. Now, B is eliminated whereupon C wins. Hence monotonicity is violated.
N.B. This example does NOT show that NER fails on monotonicity as A is elected in both profiles under NER.

Also NER fails on monotonicity when $k > 3$

30	21	20	12	12	5
C	B	A	B	A	A
A	D	B	A	C	C
D	C	D	C	B	D
B	A	C	D	D	B

The Borda scores are: A 195, B 151, C 157, D 97.
Since the average Borda score is 150, candidate D is eliminated and Borda scores are recomputed for the remaining candidates restricting the rankings to A, B and C. The scores are, respectively, 116, 86 and 98. Since the new average is 100, A wins.

Also NER fails . . ., cont'd

Suppose now that the 12 of voters with ranking $B \succ A \succ C \succ D$ had preference $A \succ B \succ C \succ D$, *i.e.* A's support would be somewhat larger. Then the Borda scores of A, B, C and D are, respectively: 207, 139, 157 and 97. Thus, both B and D are eliminated, whereupon C wins showing that NER is nonmonotonic.

Other findings about BER and NER

- there are profiles where BER exhibits the strong no-show paradox, but NER doesn't and vice versa
- there are profiles where both exhibit the strong no-show paradox while starting from the same choice in the initial profile
- neither BER or NER dominates the other when $k > 3$, i.e. we don't know which one is more likely to avoid monotonicity paradoxes

Lessons from the above

- monotonicity is one of the most compelling properties of democratic rule
- there are two primary settings of monotonicity failures:
  1. in fixed electorates
  2. in variable electorates
- there several basic types of monotonicity failure:
  1. ordinary monotonicity failure: more is less
  2. strong P-TOP: increased support for winners render them non-winners
  3. P-BOT paradox: putting an alternative last may make it a winner
- the likelihood of monotonicity failures depends on profiles
- this far only few structural characteristics of profiles have been systematically examined:
  1. the presence or absence of a Condorcet winner
  2. IC, IAC, bipolar

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