Approaches to Voting Systems

Properties, paradoxes, incompatibilities

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The main points

- many non-equivalent procedures are used for a seemingly same purpose
- all systems are based on some apparently plausible notion of winning
- each one of them has at least one major flaw
- some flaws pertain to rationality, some to fairness
- satisfaction of criteria may depend on context (input type)
- nearly all procedures can be seen as minimizing the distance between the observed profile and a system-specific goal state
- rationality of rules is expert decision settings calls for an approach that largely ignores fairness considerations

“We (all) are the champions” – or at least we can be

Five alternatives, five winners

4 voters	3 voters	2 voters
A	E	D
B	D	C
C	B	B
D	C	E
E	A	A

Table : 5 candidates, 5 winners

Plurality voting: A; plurality runoff voting: E, Condorcet extensions: D, Borda count: B; approval voting (with additional assumptions): C.

This holds for Condorcet extensions as well

. . . assuming, of course, that there is no Condorcet winner in the profile under scrutiny.

10 voters	7 voters	1 voter	7 voters	4 voters
D	B	B	C	D
A	C	A	A	C
B	A	C	B	A
C	D	D	D	B

Table : Discrepancy among some Condorcet extensions

Copeland: A, B, C
Dodgson: D
Max-min: D

Highest average ranking $\rightarrow$ Borda Count

Example

2 voters	2 voters	2 voters	1 voter
D	A	B	D
C	D	A	C
B	C	D	B
A	B	C	A

This yields the ranking DABC.
Now remove D. This gives: CBA, i.e. reversal of collective preference over A, B and C.
Fishburn: it is possible that the Borda winner wins in only one of the proper subsets of the alternative set.
Obviously, fiddling with the alternative set opens promising vistas for outcome control.

Discrepancy among positional procedures

2	2	2	3
A	A	C	D
B	D	B	B
C	C	D	C
D	B	A	A

Here the plurality winner is A, vote-for-two winner is B, vote-for-three winner is C, and the Borda winner D.

Theorem

Saari 1992. Consider the alternative set $c_1, \ldots, c_K$ of at least three elements. Then such a profile exists that alternative c_j wins when the voting rule is vote-for-j and this holds for $j = 1, \ldots, K-1$. Moreover, c_K is the Borda winner.

Does unanimity guarantee the same outcome?

No.

1 voter	1 voter	1 voter
A	A	A
B	B	B
C	C	C

Table : A unanimous profile

If vote-for-two system is used or approval voting with everyone approving of their two highest ranked alternatives, the outcome is an A-B tie, not A.

Château du Baffy experiment

number of approvals	0	1	2	3	4	5	6	7	8	9	10	> 10
number of ballots	0	2	7	3	5	2	1	1	0	0	1	0

Table : The number of approved procedures. *Source:* Laslier 2012.

Experiment, cont'd

voting rule	approvals	approving %
approval	15	68.18
alternative	10	45.45
Copeland	9	40.91
Kemeny	8	36.36
runoff	6	27.27
Coombs	6	27.27
Simpson	5	22.73
m. judgment	5	22.73
Borda	4	18.18
Black	3	13.64
range	2	9.09
Nanson	2	9.09
uncovered	1	4.54
plurality	0	0

Pairwise victories → Condorcet extensions

Example

Condorcet's paradox

4 voters	4 voters	4 voters
A	B	C
C	A	B
B	C	A

Surely, there is no winner here, or what? If so, then removing this kind of "component" from any larger profile or adding it to some profile should not change the winners, right?

Surprise?

Example

A profile with a strong Condorcet winner

7 voters	4 voters
A	B
B	C
C	A

Adding the Condorcet paradox profile to this one results in a new Condorcet winner. N.B. the Borda winner remains the same in the 11- and 23-voter profiles.

Reasons for rules

- The plurality rule has a straight-forward rationale: if only one alternative is to be chosen, it makes sense to ask each voter for his/her most preferred alternative. Whichever alternative is reported as the most preferred by more voters than any other alternative is then regarded as the social choice. Choosing any other alternative could be criticized by pointing out that the plurality winner was viewed the best by more voters than the chosen alternative.
- In some contexts the support of the majority is deemed important for legitimacy. This consideration seems to underlie the plurality runoff system which is resorted to in many elections of the head of state around the world.

Reasons, cont'd

- The motivation for electing the Condorcet winner when one exists is obvious: if an alternative defeats each of its competitors in pairwise majority comparisons, it deserves to be elected. Things get more complicated when no Condorcet winner exists. Various Condorcet extensions address this problem in different ways. Copeland's rule takes its cue in the notion of pairwise majority winning. Since the Condorcet winner defeats the largest possible $(k-1)$ other alternatives in pairwise majority comparisons, it seem plausible to use this fact in devising a system that elects the alternative that defeats more other alternatives than any of its contestants in pairwise majority comparisons.

Reasons, cont'd

- Dodgson's rule performs the smallest number of modifications (pairwise preference switches) in the observed preference profile needed to make any given alternative the Condorcet winner.
- The max-min procedure can be justified by considering the performance of each alternative when compared with its toughest competitor. If one wishes to choose the alternative that does best in this comparison, the max-min rule is an appropriate method for this purpose.

Reasons, cont'd

- The Borda count can be given two justifications. Firstly, this procedure results in the alternative that has the highest average ranking in the individual preference rankings. Secondly, by tallying the number of voters supporting a given alternative in all its $(k-1)$ pairwise comparisons one ends up with its Borda score. So, the Borda count can find its justification in the fact that the Borda winner receives the maximum number of votes when these are summed over all $k-1$ pairwise comparisons where it is present.

Some systems and performance criteria

	Criterion								
Voting system	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*
Amendment	1	1	1	1	0	0	0	0	0
Copeland	1	1	1	1	1	0	0	0	0
Dodgson	1	0	1	0	1	0	0	0	0
Maximin	1	0	1	1	1	0	0	0	0
Kemeny	1	1	1	1	1	0	0	0	0
Plurality	0	0	1	1	1	1	0	0	1
Borda	0	1	0	1	1	1	0	0	1
Approval	0	0	0	1	0	1	1	0	1
Black	1	1	1	1	1	0	0	0	0
Pl. runoff	0	1	1	0	1	0	0	0	0
Nanson	1	1	1	0	1	0	0	0	0
Hare	0	1	1	0	1	0	0	0	0

Criteria

- a: the Condorcet winner criterion
- b: the Condorcet loser criterion
- c: the strong Condorcet criterion
- d: monotonicity
- e: Pareto
- f: consistency
- g: Chernoff property
- h: independence of irrelevant alternatives
- i: invulnerability to the no-show paradox

Arrow's theorem

Theorem

Arrow'(1963): *No social welfare function satisfies the following conditions:*

1. *unrestricted domain (U)*
2. *independence of irrelevant alternatives (IIA)*
3. *Pareto (P)*
4. *non-dictatorship (D)*

Remark: social welfare functions assigns to each n-tuple of connected and transitive individual preference relations a (collective) connected and transitive preference relation.

Gibbard-Satterthwaite theorem

Definition

A social choice function is manipulable (by individuals) iff there is a situation and an individual so that the latter can bring about a preferable outcome by preference misrepresentation than by truthful revelation of his/her preference ranking, *ceteris paribus*.

Definition

A social choice function is non-trivial (non- degenerate) iff for each alternative x, there is a preference profile so that x is chosen.

Gibbard-Satterthwaite theorem, cont'd

Theorem

(Gibbard-Satterthwaite 1973-75). Every universal and non-trivial resolute social choice function is either manipulable or dictatorial.

Gärdenfors' theorem

Theorem

Gärdenfors 1976. *If a social choice function is anonymous and neutral and satisfies the Condorcet winning criterion, then it is manipulable.*

Examples of non-manipulable SCF's:

- If every voter's preference ranking is strict (no ties), then SCF that chooses the Condorcet winner when one exists and all alternatives, otherwise, is non-manipulable.
- Under the same assumption concerning voter preferences any SCF that chooses the Condorcet winner when one exists and the set of Pareto-undominated outcomes, otherwise, is also non-manipulable.

Young and Moulin

Theorem

Young: *all consistent methods are incompatible with the Condorcet winning criterion.*

Theorem

Moulin: *all procedures that satisfy the Condorcet winning criterion are vulnerable to no-show paradox.*

Theorem

Muller and Satterthwaite: *if there are at least three alternatives, then any procedure that satisfies citizen's sovereignty and (Maskin) monotonicity is dictatorial.*

The role of culture

- impartial culture: each ranking is drawn from uniform probability distribution over all rankings
- impartial anonymous culture: all profiles (i.e. distributions of voters over preference rankings) equally likely
- unipolar cultures
- bipolar cultures

Lessons from probability and simulation studies

- cultures make a difference (Condorcet cycles, Condorcet efficiencies, discrepancies of choices)
- none of the cultures mimics “reality”
- IC is useful in studying the proximity of intuitions underlying various procedures

What makes some incompatibilities particularly dramatic?

The fact that they involve intuitively plausible, “natural” or “obvious” desiderata. The more plausible etc. the more dramatic is the incompatibility.

Theorem

Moulin, Pérez: all Condorcet extensions are vulnerable to the no-show paradox.

Example

26%	47%	2%	25%
A	B	B	C
B	C	C	A
C	A	A	B

Some counterexamples are pretty hard to find: Black

Black' procedure is vulnerable to the no-show paradox, indeed, to the strong version thereof.

Example

1 voter	1 voter	1 voter	1 voter	1 voter
D	E	C	D	E
E	A	D	E	B
A	C	E	B	A
B	B	A	C	D
C	D	B	A	C

Here D is the Condorcet winner and, hence, is elected by Black. Suppose now that the right-most voter abstains. Then the Condorcet winner disappears and E emerges as the Borda winner. It is thus elected by Black. E is the first-ranked alternative of the abstainer.

Another difficult one: Nanson

5 voters	5 voters	6 voters	1 voter	2 voters
A	B	C	C	C
B	C	A	B	B
D	D	D	A	D
C	A	B	D	A

Here Nanson's method results in B.

If one of the right-most two voters abstain, C – their favorite – wins.

Again the strong version of no-show paradox appears.

The twin paradox occurs whenever a voter is better off if one or several individuals, with identical preferences to those of the voter, abstain.

Here we have an instance of the twin paradox as well: if there is only one CBDA voter, C wins. If he is joined by another, B wins.

What do we aim at?

Possible consensus states:

- consensus about everything, i.e. first, second, etc.
- consensus about the winner
- majority consensus about first rank
- majority consensus about Condorcet winner
- ...

How far are we?

Possible distance measures:

- inversion metric (Kemeny)
- discrete metric

Upshot

We have (hopefully) seen that:

- system-criterion pairs give “asymmetric” information
- only important criteria ought to be focused upon
- the likelihood of encountering problems varies with the culture
- some counterexamples are much harder to find than others

What is called for is (much) more work on structural properties of problematic profiles.

Monotonicity of Dodgson's rule

42 voters	*26 voters*	*21 voters*	*11 voters*
b	a	e	e
a	e	d	a
c	c	b	b
d	b	a	d
e	d	c	c

Alternative a needs 14 binary preference reversals to become the Condorcet winner, other alternatives need more. Hence a wins. Now, suppose that the 11 right-most voters increase the support of a by ranking it first, *ceteris paribus*. After the change, b is immediately below e in the 11-voter ranking and b needs only 9 binary preference changes to become the Condorcet winner, while a still needs 14. Therefore, the new winner is b.

Caveat

As a ranking rule the Dodgson function is nonmonotonic (Fishburn).
As a tournament aggregation rule it is monotonic.

Consistency of Kemeny's rule

As a preference function Kemeny's rule is consistent (Young and Levenglick 1978), but as a choice rule it isn't (Fishburn 1977). Choice functions map preference profiles into subsets of alternatives. Denoting by Φ the set of all preference profiles and by A the set of alternatives, we thus have

$$f : \Phi \rightarrow 2^A$$

for social choice functions.

Kemeny, cont'd

Preference functions, in contradistinction, map preference profiles into rankings over alternatives (cf. social welfare functions). I.e.

$$F : \Phi \to \mathcal{R}$$

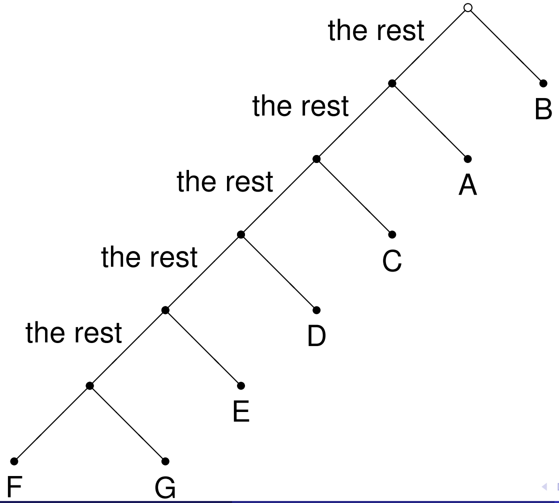
where $\mathcal{R}$ denotes the set of all preference rankings over A.

Kemeny cont'd

Consider now a partition of a set N of individuals with preference profile ϕ into two separate sets of individuals N_1 and N_2 with corresponding profiles ϕ_1 and ϕ_2 over A and assume that $f(\phi_1 \cap \phi_2) \neq \emptyset$. The social choice function f is consistent iff $f(\phi_1 \cap \phi_2) = f(\phi)$, for all partitionings of the set of individuals. The same definition can be applied to social preference functions. F is consistent iff whenever $F(\phi_1) \cap F(\phi_2) \neq \emptyset$ implies that $F(\phi_1) \cap F(\phi_2) = F(\phi)$.

It turns out that, like all Condorcet extensions, Kemeny's rule is an inconsistent social choice function. An example is provided by Fishburn (1977, 484). However, as a preference function it is consistent.

A successive agenda



An amendment agenda

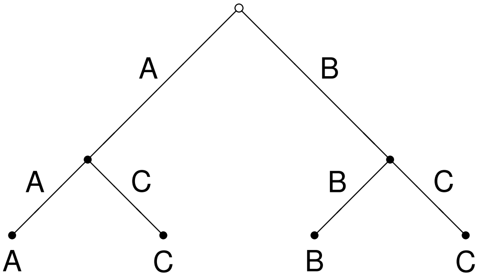


Figure : An amendment agenda

Results on agenda systems I

- Condorcet losers are not elected (not even under sincere voting),
- sophisticated voting avoids the worst possible outcomes, i.e those outside the Pareto set
- Condorcet winner is elected (even under sincere voting) by the amendment procedure,
- the strong Condorcet winner is elected by both systems.

Results on agenda systems II

- McKelvey's (1979) results on majority rule and agenda-control.
- All Condorcet extensions are vulnerable to the no-show paradox (Moulin 1988, Pérez 2001).
- Pareto violations are possible
- successive procedure is more vulnerable to agenda manipulation than the the amendment procedure (Barberà and Gerber 2017)
- of all quota rules, the simple majority rule maximizes the set of profiles that are not manipulable for both successive and amendment procedures (Barberà and Gerber 2017)

How about tournaments?

- rankings just aren't always plausible
- individual decision making with multiple criteria
- best variant choice problems
- much background work is already available

All ranking profiles can be mapped into tournaments

Example

4	3	2
A	B	C
B	C	B
C	A	A

$\Longrightarrow$

	A	B	C
A	-	4	4
B	5	-	7
C	5	2	-

$\Longrightarrow$

	A	B	C
A	-	0	0
B	1	-	1
C	1	0	-

Remark

What we have above is a simple majority tournament.

Remark

Ties call for special arrangements, e.g. $\frac{1}{2}$ *points to each element.*

. . . and all tournaments into profiles

Theorem

McGarvey 1953. Given an arbitrary preference pattern [relation], over a set of n elements, a group of individuals exists with strong individual preference orderings [complete, asymmetric and transitive] such that the group preference pattern as determined by the method of simple majority decision is the given preference pattern.

How many voters are needed?

Thus, with k alternatives, there are $k(k-1)/2$ pairwise comparisons. Consequently, this is the maximum number of voters one needs to generate a preference profile that translates into any given tournament. Is this also the minimum? No, says McGarvey:

> *. . . the actual minimum number of individuals necessary to express all possible patterns over n elements has not been ascertained, but we conjecture that it is approximately n.*

How many, cont'd

Theorem

Stearns 1959. The number of voters need not be larger than $k + 2$.

Theorem

Knoblauch 2013. Head-to-head (absolute) majority membership voting with voters having complete and transitive preferences can implement an arbitrary binary relation over the set of alternatives. The number of voters can be chosen to be smaller than $4 \times k - 2$.

How many, cont'd

Remark

Head-to-head membership voting defines a binary relation V over alternatives as follows:

$$xVy \Leftrightarrow |\{v \in N | x \succ y\}| > |N|/2.$$

Remark

In Knoblauch's theorem, the collective preference relation does not have to be complete. Hence, it is a generalization of McGarvey and Stearns.

Slater's rule

- given a set of k alternatives, generate all $k!$ rankings
- convert these into tournaments
- measure the distance between these and the individual tournaments
- pick the closest one(s): the underlying ranking is the solution

Ambiguity

Stob's example:

	A	B	C	D	E	F	G	score
A	0	0	1	1	1	1	1	5
B	1	0	1	1	0	0	0	3
C	0	0	0	1	1	1	0	3
D	0	0	0	0	1	1	1	3
E	0	1	0	0	0	1	1	3
F	0	1	0	0	0	0	1	2
G	0	1	1	0	0	0	0	2

Two Slater rankings: $B \succ A \succ C \succ D \succ E \succ F \succ G$ and $A \succ C \succ D \succ E \succ F \succ G \succ B$.
N.B. B is ranked *first or last* in these two rankings.

Discrepancies

- the Slater winner may be in an position in Dodgson ranking (Klamler)
- the Slater and Copeland rankings can be far from each other (Charon and Hudry)
- prudent order (Arrow and Raynaud) may be exact opposite of the Slater ranking (Lamboray)
- the unique Slater winner may be in any position of a prudent order (Lamboray)
- the Banks and Slater sets can be disjoint when the number of alternatives is at least 14 (Östergård and Vaskelainen)

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